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Dynamical Blume–Capel Model: Competing Metastable States at Infinite Volume

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This paper concerns the microscopic dynamical description of competing metastable states. We study, at infinite volume and very low temperature, metastability and nucleation for kinetic Blume–Capel model: a ferromagnetic lattice model with spins taking three possible values: $-1, 0, 1$. In a previous paper ([MO]) we considered a simplified, irreversible, nucleation-growth model; in the present paper we analyze the full Blume–Capel model. We choose a region U of the thermodynamic parameters such that, everywhere in U: $-\underline{1}$ (all minuses) corresponds to the highest (in energy) metastable state, $\underline{0}$ (all zeroes) corresponds to an intermediate metastable state and $+\underline{1}$ (all pluses) corresponds to the stable state. We start from $-\underline{1}$ and look at a local observable. Like in [MO], we find that, when crossing a special line in U, there is a change in the mechanism of transition towards the stable state $+\underline{1}$. We pass from a situation:

1. where the intermediate phase $\underline{0}$ is really observable before the final transition, with a permanence in $\underline{0}$ typically much longer than the first hitting time to $\underline{0}$; to the situation:

2. where $\underline{0}$ is not observable since the typical permanence in $\underline{0}$ is much shorter than the first hitting time to $\underline{0}$ and, moreover, large growing 0-droplets are almost full of $+1$ in their interior so that there are only relatively thin layers of zeroes between $+1$ and -1.

KEY WORDS: metastability; nucleation; Blume–Capel model; reversibility.

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1. INTRODUCTION

In this paper we analyze metastability and nucleation at infinite volume and very low temperature, in the framework of the dynamical Blume–Capel model in two dimensions.

The Blume–Capel model describes a ferromagnetic lattice spin system where the single spin variable can take three possible values: $-1, 0, +1$.

The formal hamiltonian is given by:

$$H(\sigma) = \sum_{\langle x, y \rangle} (\sigma(x) - \sigma(y))^2 - \lambda \sum_x \sigma(x)^2 - h \sum_x \sigma(x) \tag{1.1}$$

where λ and h are two real parameters, having the meaning of the chemical potential and the external magnetic field, respectively and $\langle x, y \rangle$ denotes a generic pair of nearest neighbor sites in $\mathbb{Z}^2$.

In a previous paper; [MO], we considered a simplified version given by an irreversible, nucleation-and-growth model similar to that introduced by Dehghanpour and Schonmann in [DSch1] as a simplified version of stochastic Ising model.

We refer to [MO] for:

1. A general introduction to metastability.

2. A discussion of "competing metastable states."

3. A comparison between the finite volume and infinite volume metastable behavior at low temperature.

In the present paper we extend the results of [MO] to the full Blume–Capel model.

Our setup will correspond to the one considered in [DSch2] in the case of stochastic Ising model: an infinite volume system with given λ, h in the limit of zero temperature.

In [CiO] metastability and nucleation were studied for the two-dimensional Blume–Capel model in a different setup: the so-called Freidlin–Wentzell asymptotic regime: system enclosed in a finite torus Λ, with given λ, h, in the limit of zero temperature. The interesting region of parameters that was analyzed in [CiO] is

$$0 < |\lambda| < h \tag{1.2}$$

Let $-\underline{1}$, $\underline{0}$ and $+\underline{1}$ denote the configurations with all spins equal to $-1, 0, +1$, respectively. It is easy to see that for small enough λ and h, these three configurations are local minima for the energy.

In the whole region (1.2) we have (in any finite volume):

$$H(-\underline{1}) > H(\underline{0}) > H(+\underline{1})$$

In other words $-\underline{1}$ and $\underline{0}$ are metastable whereas $+\underline{1}$ is stable. The main question that arises is the following: when starting from the highest metastable configuration $-\underline{1}$, whether or not the system reaches the intermediate metastable configuration $\underline{0}$ before relaxing to the stable configuration $+\underline{1}$.

In [CiO] the answer was found in the Freidlin–Wentzell scenario. It has been shown, in [CiO], that there is a change in the mechanism of transition from $-\underline{1}$ to $+\underline{1}$ when crossing the line $h = 2\lambda > 0$ in the h, λ plane. On the right side of this line ($h < 2\lambda$) the transition is "direct" i.e., the system goes to $+\underline{1}$ via the formation of a special critical droplet: a squared "picture frame" with suitably large size, made by a square of pluses encircled by a unit layer of zeroes in a sea of minuses. It is easy to see that a direct interface between minus and plus is unstable in our region of parameters. This explains the persistence of a thin layer of zeroes between the pluses and the minuses. In the other region ($h > 2\lambda$) the transition is "indirect" in the sense that the system first reaches the $\underline{0}$ configuration via the formation of a supercritical squared droplet of zeroes in a sea of minuses. Subsequently, via the formation of a supercritical squared droplet of pluses in a sea of zeroes, the system is driven to the final stable state $+\underline{1}$. In this latter region ($h > 2\lambda$) the two transitions are "Ising-like" whereas in the former region ($h < 2\lambda$) the mechanism of transition and in particular the associated interface dynamics are much more complicated.

It is natural to pose the problem of the behavior of the kinetic two-dimensional Blume–Capel model in infinite volume and, in particular, in the Dehghanpour–Schonmann regime ($\beta \to \infty$ for small but fixed λ and h in infinite volume). In particular it is natural to pose the following question: does the sort of "dynamical phase transition" that has been detected in finite volume persists in infinite volume? If yes in which form? One easily realizes that it is reasonable to expect a change in the mechanism of transition over the line $\lambda = 0$. Indeed, when passing from $\lambda < 0$ to $\lambda > 0$ two simultaneous effects take place:

1. The local energy barrier, strictly related to the nucleation rate, between $-\underline{1}$ and $\underline{0}$ becomes higher than the local energy barrier between $\underline{0}$ and $+\underline{1}$.

2. The speed of growth of a supercritical droplet of pluses in a sea of zeroes becomes larger than the speed of growth of a supercritical droplet of zeroes in a sea of minuses.

In other words one expects that, starting from $-\underline{1}$ and looking at an observable localized close to the origin, for $\lambda < 0$ one first sees a large droplet of zeroes coming from a large distance and after a much larger time one observes the arrival of a large droplet of pluses; on the contrary, for $\lambda > 0$ the time of the first arrival of the zero phase near the origin is much longer than the time interval needed for the subsequent arrival of the pluses.

We indeed prove (Theorem 1) a change in the asymptotic behavior of the ratio between the time of first appearance, say, of a stable non-minus situation at the origin, denoted by $\tau_{\oplus}$, and the time interval, denoted by $\tau_{\oplus +}$, between $\tau_{\oplus}$ and the first appearance of the $+$ phase at the origin. When $\tau_{\oplus +} \ll \tau_{\oplus}$, we also give information on the shape of large droplets: we show, in Theorem 2, that large droplets of zeroes tend to be invaded by pluses in their interior so that, asymptotically, they become completely full of pluses with only a relatively thin layer of non-minuses (typically zeroes) between the internal pluses and the sea of minuses.

The rest of the paper is organized as follows: in Section 2 we give definitions and preliminary results. In Section 3 we give the main results. In particular we state Proposition 3 concerning upper and lower estimates for $\tau_{\oplus}$, $\tau_{\oplus +}$ from which Theorem 1 immediately follows. In Section 4 we extend the results about exit times for Metropolis Markov chains to study the exit probability at "small times." In Section 5, we carry out a preliminary analysis of the dynamics. In Section 6 we analyze the first metastable regime and prove Proposition 3,a). In section 7 we analyze the relaxation to non-minuses and prove Proposition 3,b). In Section 8 and 9 we prove Proposition 3,c) and Proposition 3,d), respectively. In Section 10 we prove Theorem 2.

2. DEFINITIONS AND PRELIMINARY RESULTS

2.1. Basic Notation

By

$$\lfloor a \rfloor \qquad \text{and} \qquad \lceil a \rceil \tag{2.1}$$

we denote the largest integer smaller than or equal to the real number a and the smallest integer larger than or equal to a, respectively.

For $a \in \mathbb{R}$, we denote by

$$[a]^+ := \max\{a, 0\} \tag{2.2}$$

the *positive part* of a.

For $x \in \mathbb{Z}^2$,

$$\|x\| := \max_{i=1,2} |x_i| \tag{2.3}$$

$$\|x\|_1 := |x_i| + |x_2| \tag{2.4}$$

For $B \subset \mathbb{Z}^2$,

$$\operatorname{diam}(B) := \sup_{x, y \in B} \|x - y\| \tag{2.5}$$

Given a subset A of a space Ω where a notion of neighbor is given, we define the following boundary sets:

$$\partial A \equiv \partial^+ A := \{\xi \in \Omega \setminus A : \exists \eta \in A \text{ s.t. } \xi \text{ and } \eta \text{ are neighbors}\} \tag{2.6}$$

$$\partial^- A := \partial^+ A^c \tag{2.7}$$

where $A^c := \Omega \setminus A$.

2.2. The Model

The configuration space is $S := \{-1, 0, 1\}^{\mathbb{Z}^2}$; the formal hamiltonian is given in (1.1). Given a configuration $\sigma \in S$, $\sigma(x) \in \{-1, 0, 1\}$ is called *spin* at site $x \in \mathbb{Z}^2$ and $\langle x, y \rangle$ denotes a pair of nearest neighbor sites in $\mathbb{Z}^2$.

We recall that $-\underline{1}$, $\underline{0}$ and $+\underline{1}$ denote the configurations with all spins equal to $-1, 0, +1$, respectively. Consider a Blume– Capel system enclosed in a finite torus Λ (square with periodic boundary conditions). The zero-temperature phase diagram for different values of our parameters λ and h is the following one:

for $\lambda = h = 0$,	the ground state is three times degenerate, the configurations minimizing the energy are $-\underline{1}$, $\underline{0}$ and $+\underline{1}$;
for $h > 0$ and $h > -\lambda$,	the ground state is $+\underline{1}$;
for $h < 0$ and $h < \lambda$,	the ground state is $-\underline{1}$;
for $\lambda < 0$ and $\lambda < h < -\lambda$,	the ground state is $\underline{0}$.

For $h = 0$, $\lambda > 0 : +\underline{1}$, $-\underline{1}$ coexist. For $h = \lambda < 0 : -\underline{1}$, $\underline{0}$ coexist. For $h = -\lambda > 0 : +\underline{1}$, $\underline{0}$ coexist. These results are summarized in Fig. 1 where the coexistence lines are shown.

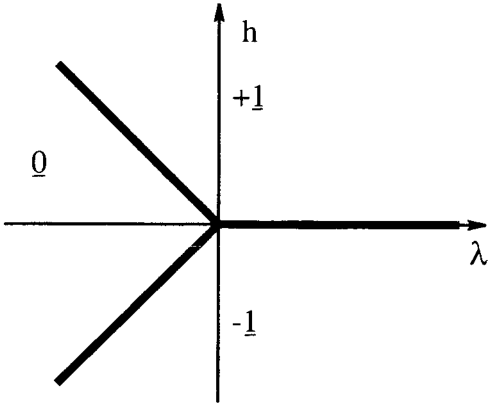


Fig. 1. Phase diagram at zero temperature.

We write

$$\oplus := \{\eta : \forall\, x \in \Lambda\; \eta(x) \neq -1\} \tag{2.8}$$

to denote the set of all configurations free of minuses.

We say that two configurations σ, σ' are *nearest neighbors* if they differ only for the value of the spin at a single site x and the absolute value of the spin variation in x is one. The corresponding increment in energy is given by:

$$\begin{aligned} \Delta H(\sigma, x, \sigma(x), \sigma'(x)) &= (4-\lambda)\, \Delta\sigma^2(x) - (2(n_x^+ - n_x^-) + h)\, \Delta\sigma(x) \\ &= \lambda'\, \Delta\sigma^2(x) - h'\, \Delta\sigma(x) \end{aligned} \tag{2.9}$$

where $\Delta\sigma^2(x) := \sigma'(x)^2 - \sigma(x)^2$, $\Delta\sigma(x) := \sigma'(x) - \sigma(x)$ and n_x^+ or n_x^- is the number of nearest neighbors of the site x with spin $+1$ or -1, respectively.

$\lambda' := 4 - \lambda$ and $h' = h'_x(\sigma) := (2(n_x^+ - n_x^-) + h)$ have the meaning of effective chemical potential and magnetic field, respectively. Formally, $\Delta H = H(\sigma') - H(\sigma)$.

We write

$$\sigma \geqslant \sigma' \qquad \text{if} \quad \sigma(x) \geqslant \sigma'(x) \qquad \forall\, x \in \mathbb{Z}^2 \tag{2.10}$$

We will repeatedly make use of the fact that h' is an increasing function of σ w.r.t. the partial ordering given by (2.10).

The dynamics is given by a continuous time Markov process defined by the following Metropolis transition rates from the configuration σ to the configuration $\sigma' \neq \sigma$:

$$c_\beta(\sigma, \sigma') := \begin{cases} e^{-\beta[\Delta H(\sigma, x, \sigma(x), \sigma'(x))]^+} & \text{if } \sigma \text{ and } \sigma' \text{ are n.n. with } \sigma(x) \neq \sigma'(x) \\ 0 & \text{otherwise} \end{cases} \tag{2.11}$$

The parameter β has the meaning of inverse temperature.

It is immediate to verify that the rates (2.11) fulfill the reversibility condition w.r.t. the Gibbs measure corresponding to the hamiltonian (1.1) (see [Li]).

Two configurations σ and σ' are called *connected* if $c_\beta(\sigma, \sigma') \neq 0$. With our dynamics, two configurations are connected if and only if they are nearest neighbors.

Note that here (unlike the dynamics used in [CiO] to study metastability for the Blume–Capel model in finite volume) direct transitions from -1 to $+1$ and vice versa are not allowed.

We will show in the next subsection that with this dynamics we can take advantage of the attractivity of the hamiltonian.

It is easy to see that in the large β regime this dynamics is equivalent to that in [CiO] in the sense that all results obtained there can be transported to our case. However, since we look specifically at the $-\underline{1} \to \oplus$ transition (which was not considered in [CiO]) and since we are interested in the whole region of parameters $h > |\lambda|$ (in [CiO] $\lambda < 0$ was only partially discussed), we analyze in Subsection 6.1 the energy landscape and in particular the saddles in a self- contained way.

We show that in finite volume and for sufficiently large β typical decay times and patterns from the metastable state $-\underline{1}$ to the stable state $+\underline{1}$ are the same in both dynamics (see Lemmata 6.2, 6.3, below).

2.3. Basic Coupling

In order to exploit the attractive (ferromagnetic) nature of the hamiltonian, we introduce a coupling between processes starting from different configurations and possibly evolving in different volumes and feeling different boundary conditions: we associate with every site $x \in \mathbb{Z}^2$, independently from site to site, a sequence of i.i.d. exponentially-distributed random variables with rate 1. We denote these sequences by $\{t_{x,i}^{-0}\}_{i \in \mathbb{N}}$, $\{t_{x,i}^{0-}\}_{i \in \mathbb{N}}$, $\{t_{x,i}^{0+}\}_{i \in \mathbb{N}}$ and $\{t_{x,i}^{+0}\}_{i \in \mathbb{N}}$.

Let be $\eta \in \{-1, 0, +1\}^{\mathbb{Z}^2}$ and $b \in \{-1, 0, +1\}$; we denote by $\eta^{b,x}$ the configuration obtained from η by setting to the value b the spin in x and leaving the spin of the other sites unchanged. Let us now define the updating rule for the process σ_t: At time t_x^{ab}:

1. $\sigma(y)$ with $y \neq x$, remains unchanged.

2. if $\sigma(x) \neq a$, σ remains unchanged, otherwise, if $\sigma(x) = a$, we extract a random variable u, uniformly distributed in $[0, 1]$ independently of any other variable; if $u < c_\beta(\sigma, \sigma^{b,x})$, then we set $\sigma_{t_x^{ab}}(x) = b$ namely, $\sigma_{t_x^{ab}} = \sigma^{b,x}$.

The general expression of the finite-volume hamiltonian in the volume $\Lambda \subset \mathbb{Z}^2$ is given by:

$$H^\zeta_\Lambda(\sigma) := \sum_{\langle x,y\rangle \in \Lambda} (\sigma(x) - \sigma(y))^2 - h \sum_{x \in \Lambda} \sigma(x) - \lambda \sum_{x \in \Lambda} \sigma(x)^2$$
$$+ \sum_{\substack{\langle x,y\rangle \text{ s.t.} \\ x \in \Lambda,\, y \in \Lambda^c}} (\sigma(x) - \zeta(y))^2 \tag{2.12}$$

where $\sigma_\Lambda \in \{-1, 0, +1\}^\Lambda$ and $\zeta \in \{-1, 0, +1\}^{\Lambda^c}$ is the boundary condition.

We define the finite volume dynamics corresponding to different initial and/or boundary conditions on the same probability space: We call

$$\sigma^\eta_{\Lambda;\zeta;t} \tag{2.13}$$

the process in the volume Λ, with boundary condition ζ and initial condition η. To construct the random configuration $\sigma^\eta_{\Lambda;\zeta;t}$ we only use the random times "internal" to Λ and the rates $c_\beta(\sigma, \sigma') = c^\zeta_\beta(\sigma, \sigma')$ computed according to (2.11) with $\Delta H(\sigma, \sigma', a, b) = H^\zeta_\Lambda(\sigma') - H^\zeta_\Lambda(\sigma)$. We will omit Λ, η or ζ from notation if $\Lambda = \mathbb{Z}^2$, $\eta = -\underline{1}$ or $\zeta = -\underline{1}$, respectively.

Given a square Λ, on the probability space of Poisson times, we also introduce the finite volume process under periodic boundary conditions

$$\bar\sigma^\eta_{\Lambda;t}$$

Notice that simultaneous spin changes have zero probability; we will often implicitly use this fact in the following proofs.

A fundamental property of the above defined coupling is that it preserves the partial ordering (2.10) among the configurations.

If $\eta \leqslant \eta'$, $\zeta \leqslant \zeta'$, and $\Lambda \subseteq \Lambda'$, then

$$\begin{aligned} \sigma^\eta_{\Lambda;\zeta;t} &\leqslant \sigma^{\eta'}_{\Lambda;\zeta';t} \\ \sigma^\eta_{\Lambda;t} &\leqslant \sigma^{\eta'}_{\Lambda';t} \\ \sigma^\eta_{\Lambda;t} &\leqslant \sigma^{\eta'}_{t} \\ \sigma^\eta_{\Lambda;t} &\leqslant \bar\sigma^\eta_{\Lambda;t}. \end{aligned} \tag{2.14}$$

The key point is that, since the spin jump is not larger than one, it is impossible for the lower process to overtake the higher one.

As an example, we prove the first inequality in (2.14): Let $s = t_x^{ab}$ be the first time at which, ab absurdo, $\sigma^{\eta'}_{\Lambda;\zeta';s}(x) < \sigma^{\eta}_{\Lambda;\zeta;s}(x)$. Let $s' < s$ be the last time before s at which there is a spin change and let $\bar{s} \in]s', s[$. Since the dynamics does not allow a spin change larger than one, $\sigma^{\eta}_{\Lambda;\zeta;\bar{s}}(x)\ \sigma^{\eta'}_{\Lambda;\zeta';\bar{s}}(x)$. It would follow $c_\beta(\sigma^{\eta}_{\Lambda;\zeta;\bar{s}}, \sigma^{\eta}_{\Lambda;\zeta;s}) > c_\beta(\sigma^{\eta'}_{\Lambda;\zeta';\bar{s}}, \sigma^{\eta'}_{\Lambda;\zeta';s})$. This is absurd by (2.9) and (2.11), since $h'_x(\sigma)$ is an increasing function of σ.

We stress that for the dynamics used in [CiO], it is not possible to define a coupling with the above mentioned properties. This is the main reason why we are using a different (though equivalent) dynamics.

It is possible to relate our continuous-time process in a finite volume Λ to a discrete time version given by the Markov chain $X_j := \sigma_{\Lambda; t_j}$, where $t_0 := 0$ and t_{j+1} is the first time after t_j when a Poisson clock $t_x^{a,b}$ "rings" inside Λ. We are particularly interested in hitting times to given sets of configurations. These times will turn out to be typically exponentially long in β. By using standard large-deviation estimates on Poisson times, it is easy to show that there is no difference between the continuous time and the discrete time dynamics as far as large β asymptotics of these times is concerned. Therefore, we will immediately transport to the continuous-time case all results given in [OS] for the discrete-time, finite-volume dynamics.

Let us consider a finite box Λ with periodic boundary conditions. Given a subset $\mathcal{O}$ of the configuration space, it is interesting to consider the process

$$\tilde{\sigma}^{\mathcal{O};\eta}_{\Lambda;t} \tag{2.15}$$

restricted to $\mathcal{O}$. It is defined in the following way: starting from $\eta \in \mathcal{O}$, we update $\tilde{\sigma}^{\mathcal{O};\eta}_{\Lambda;t}$ with the same rule as $\bar{\sigma}^{\eta}_{\Lambda;t}$ whenever the Poisson marks do not lead the process $\tilde{\sigma}^{\mathcal{O};\eta}_{\Lambda;t}$ out of $\mathcal{O}$; we simply ignore the Poisson marks associated to the moves that bring the process outside $\mathcal{O}$ (reflecting barrier on $\partial\mathcal{O}$). In particular, we will use the finite volume processes restricted to $\{-1, 0\}^\Lambda$ and to $\{0, 1\}^\Lambda$, we call them

$$\mathscr{I}^- \quad \text{and} \quad \mathscr{I}^+ \tag{2.16}$$

respectively. By looking at the transition rates, it is easy to notice that these processes are in fact dynamic Ising models with magnetic field $h-\lambda$ and $h+\lambda$, respectively. We will use these processes (and the results about the Ising model obtained in [DSch2]) to bound from above and from below $\bar{\sigma}^{\eta}_{\Lambda;t}$.

2.4. Critical Lengths and Activation Energies

We introduce some interesting side-lengths useful to characterize the tendency of droplets to grow or shrink:

$$L^* := \frac{2}{h+\lambda} \qquad \tilde{L} := \frac{2+2\lambda}{h+\lambda}$$
$$M^* := \frac{2}{h-\lambda} \qquad \tilde{M} := \frac{2-2\lambda}{h-\lambda} \tag{2.17}$$
$$\ell^* := \frac{2-h+\lambda}{h}$$

We introduce the following energies:

$$\begin{aligned} \Gamma_{\oplus} := \min \{ & 8\lceil \ell^* \rceil + 8 - (2\lceil \ell^* \rceil^2 + 3\lceil \ell^* \rceil + 5)\, h + (5\lceil \ell^* \rceil + 3)\, \lambda, \\ & 8\lceil \ell^* \rceil + 4 - (2\lceil \ell^* \rceil^2 + \lceil \ell^* \rceil - 4)\, h + 5\lceil \ell^* \rceil\, \lambda, \\ & 4\lceil M^* \rceil - (h-\lambda)(\lceil M^* \rceil^2 - \lceil M^* \rceil + 1)\} \\ = \min & \left\{ \frac{8+2h+10\lambda}{h}, \frac{4+2h-2\lambda}{h-\lambda} \right\} + O(h+\lambda). \end{aligned} \tag{2.18}$$

$$\Gamma_{+} := 4\lceil L^* \rceil - (h+\lambda)(\lceil L^* \rceil^2 - \lceil L^* \rceil + 1) = \frac{4+2h+2\lambda}{h+\lambda} + O(h+\lambda) \tag{2.19}$$

We will look at $\Gamma_{\oplus}$ as the activation energy of a critical droplet of zero-pluses and Γ_{+} as an upper estimate to the activation energy of the critical droplet of pluses in the sea of "non-pluses" (in fact it is the Ising-like activation energy of a critical droplet of pluses in the sea of zeroes).

Note that, as it has been shown in [CiO], in the region $0<\lambda<h<2\lambda$, in finite volume the pluses are directly created from the minus phase without passing through the configuration $\underline{0}$ and therefore there is no need to consider Γ_{+}. However, we will see that Γ_{+} provides a useful upper estimate for our infinite- volume results.

The following quantities represent the energies associated to the unit-square protuberances of zeroes or pluses respectively and they characterize the speeds of local growth of the zeroes and pluses:

$$\gamma_{\oplus} \equiv \gamma_0 := 2-h+\lambda \tag{2.20}$$

$$\gamma_{+} := 2-h-\lambda \tag{2.21}$$

It will turn out that the typical hitting times to the origin for the non-minuses and the pluses are of the form $\ell^{\beta\kappa}$, where k for different values of λ and h can be given by:

$$k_0 := \frac{\Gamma_{\oplus} + \gamma_0}{3} \tag{2.22}$$

$$k_+ := \frac{\Gamma_+ + \gamma_+}{3} \tag{2.23}$$

$$k_* := \frac{\Gamma_+ + \max\{\gamma_0, \gamma_+\}}{3} \tag{2.24}$$

In the present paper, we will only consider the region $h > |\lambda| > 0$, where $H(-\underline{1}) > H(\underline{0}) > H(+\underline{1})$.

We detect a change in the mutual relationships between the relevant quantities (activation energies) when crossing the $\lambda = 0$ line:

1. if $h > -\lambda > 0$,

$$\Gamma_+ > \Gamma_{\oplus}, \qquad \gamma_+ > \gamma_0, \qquad k_+ = k_* > k_0 \tag{2.25}$$

2. if $h > \lambda > 0$,

$$\Gamma_+ < \Gamma_{\oplus}, \qquad \gamma_+ < \gamma_0, \qquad k_+ < k_* < k_0 \tag{2.26}$$

The main result of this paper (see Theorem 1) is that this change is related to a sort of dynamical phase transition at $\lambda = 0$.

We will use the following symbols:

$$\mathbb{P}_\eta(\mathscr{E}) := \mathbb{P}(\mathscr{E} \mid \sigma_0 = \eta) \tag{2.27}$$

is the probability of the event $\mathscr{E}$ (set of trajectories) starting from the configuration η.

We denote by

$$Q(L) \tag{2.28}$$

the square with side-length $2L+1$ centered at the origin,

$$\Lambda^0 := Q(\lfloor e^{\beta(k_0 - (\gamma_0/2))} \rfloor) \tag{2.29}$$

$$\Lambda^* := Q(\lfloor e^{\beta(k_* - (\gamma_0/2))} \rfloor) \tag{2.30}$$

$$\bar{\Lambda} := Q(\lfloor e^{\beta k_0} \rfloor) \tag{2.31}$$

For technical reasons, we introduce two sufficiently large constants (independent of β) D and D'.

$$Q := Q(D) \tag{2.32}$$

The two constants D and D' are chosen large enough to ensure that 1) the "critical droplet" in the volume Q has the same shape as the "critical droplet" in infinite volume (see Subsection 6.1 and Lemma 7.1); 2) that the nucleation rate in Q is not influenced by the spins outside Q (see Lemma 6.6) and 3) that once the volume Q is completely full of non-minuses (or pluses) it is very unlikely that too many minuses (or nonpluses) will form thereafter (see Lemmata 7.2, 9.3). We take $D' = D/5$.

2.5. The Infection Processes

We will be interested in analyzing the $-\underline{1} \to \oplus$ and the $\oplus \to +\underline{1}$ transitions.

For $x \in \mathbb{Z}^2$, we recursively define the following $\oplus$-*disinfection* times $\hat{\tau}_i^{\oplus}(x)$ and $\oplus$-infection times $\check{\tau}_i^{\oplus}(x)$:

$$\hat{\tau}_0^{\oplus}(x) := 0$$

$$\check{\tau}_i^{\oplus}(x) := \inf\{t > \hat{\tau}_{i-1}^{\oplus}(x) \text{ such that the number of minuses in } Q+x \text{ is } 0\}$$

$$\hat{\tau}_i^{\oplus}(x) := \inf\left\{t > \check{\tau}_i^{\oplus}(x) \text{ s.t. the number of minuses in } Q+x \text{ is at least } \frac{D}{3}\right\}$$

A site x is called $\oplus$-infected for the process $\sigma_t(x)$ at time t if $t \in [\check{\tau}_i^{\oplus}(x), \hat{\tau}_i^{\oplus}(x)[$, for some $i > 0$.

In the same way, we introduce the $+$*infection* and $+$*disinfection* times $\check{\tau}_i^{+}(x)$ and $\hat{\tau}_i^{+}(x)$, but here we consider the volume $Q(D-1)$ and give conditions on the number of "non-pluses" instead of the number of minuses:

$$\hat{\tau}_0^{+}(x) := 0$$

$$\check{\tau}_i^{+}(x) := \inf\{t > \hat{\tau}_{i-1}^{+}(x) \text{ such that the number of non-pluses in } Q(D-1)+x \text{ is } 0\}$$

$$\hat{\tau}_i^{+}(x) := \inf\left\{t > \check{\tau}_i^{+}(x) \text{ s.t. the number of non-pluses in } Q(D-1)+x \text{ is at least } \frac{D}{3}\right\}$$

The quantities that we want to estimate are

$$\tau_{\oplus} := \check{\tau}_1^{\oplus}(0) \tag{2.33}$$

$$\tau_{+} := \check{\tau}_1^{+}(0) \tag{2.34}$$

and

$$\tau_{\oplus +} := \tau_{+} - \tau_{\oplus} \tag{2.35}$$

Note that with this definition, the infection time unfortunately depends on D. However, this dependence will turn out to be irrelevant for the results we give. Indeed, the key finite-volume results of Lemma 4.3 guarantee that the time needed to reach the stable configuration has the same asymptotic behavior, for large β, as the time needed to reach the critical droplet and thus it does not depend significantly on D as far as D is sufficiently large but independent of β.

We say that the process $\sigma_{\Lambda^0;t}$ *Locally Exceeds the Critical Energy* (LECE) at time t if there is some translate Q' of Q inside Λ^0 such that $H_{Q'}(\sigma_{Q';t}) - H_{Q'}(-\underline{1}) > \Gamma_{\oplus}$.

We define the following stopping time, having the meaning of nucleation time in Λ^0:

$$\tau_{\text{LECE}}^{\Lambda^0} := \min\{t : \sigma_{\Lambda^0;t}\ \text{LECE}\} \tag{2.36}$$

2.6. Space-Time Clusters

Two space-time points (x, t) and (y, t') with $t < t'$ are called *directly zero-plus connected ($\oplus$-connected)* if either $\{x = y, \sigma^{\eta}_{\Lambda;\zeta;s}(x) \neq -1\ \forall s \in [t, t']\}$ or $\{t = t', \|x - y\| = 1, \sigma^{\eta}_{\Lambda;\zeta;t}(x) \neq -1 \text{ and } \sigma^{\eta}_{\Lambda;\zeta;t}(y) \neq -1\}$.

Two space-time points (x, t) and (y, s) are called *$\oplus$-connected* if there exists a sequence of pairwise directly $\oplus$-connected points starting from (x, t) and ending in (y, s).

Consider the process σ at times up to time t. Let (x, t') be a space-time point. We call *space-time cluster of zero-pluses*, the maximal $\oplus$-connected set $\mathscr{C}^{\oplus}_{x,t'}(\sigma, t)$ of space-time points (y, s) such that $s \leqslant t$, that contains (x, t').

Likewise, two space-time points (x, t) and (y, t') with $t < t'$ are called *directly $+$connected* if either $\{x = y, \sigma^{\eta}_{\Lambda;\zeta;s}(x) = 1\ \forall s \in [t, t']\}$ or $\{t = t', \|x - y\| = 1, \sigma^{\eta}_{\Lambda;\zeta;t}(x) = 1 \text{ and } \sigma^{\eta}_{\Lambda;\zeta;t}(y) = 1\}$.

Two space-time points (x, t) and (y, s) are called *$+$connected* if there exists a sequence of pairwise directly $+$connected points starting from (x, t) and ending in (y, s).

We call *space-time cluster of pluses*, the maximal +connected set $\mathscr{C}^{+}_{x,t'}(\sigma, t)$ of space- time points (y, s) such that $s \leqslant t$, that contains (x, t').

We call *section at time t* of a space-time cluster the set of all sites x such that (x, t) is in the space-time cluster (note that this set may be non-connected).

The *width* $|\mathscr{C}(\sigma, t)|$ of a space-time cluster is defined to be the largest spatial distance $\|x-y\|$ between the points (x, t), (y, s) in the space-time cluster.

We will often use the terms *cluster* or *droplet* for a *-cluster (in the sense of site percolation, see [G]).

3. MAIN RESULTS

Our main results are contained in the following theorems:

Theorem 1. $\exists h_0$ such that $\forall\, \delta > 0$,

1. In the region $0 < -\lambda < h < h_0$ (where $\Gamma_+ > \Gamma_\oplus$ and $\gamma_+ > \gamma_0$),

$$1.1)\quad \mathbb{P}(e^{\beta((\Gamma_\oplus + \gamma_0)/3-\delta)} < \tau_\oplus < e^{\beta((\Gamma_\oplus + \gamma_0)/3+\delta)}) \xrightarrow{\beta\to\infty} 1$$

$$1.2)\quad \mathbb{P}(e^{\beta((\Gamma_+ + \gamma_+)/3-\delta)} < \tau_+ < e^{\beta((\Gamma_+ + \gamma_+)/3+\delta)}) \xrightarrow{\beta\to\infty} 1$$

$$1.3)\quad \mathbb{P}\left(\frac{\tau_{\oplus +}}{\tau_\oplus} > e^{\beta((\Gamma_+ - \Gamma_\oplus + \gamma_+ - \gamma_0)/3-\delta)}\right) \xrightarrow{\beta\to\infty} 1$$

2. In the region $0 < \lambda < h < h_0$ (where $\Gamma_\oplus > \Gamma_+$ and $\gamma_0 > \gamma_+$),

$$2.1)\quad \mathbb{P}(e^{\beta((\Gamma_\oplus + \gamma_0)/3-\delta)} < \tau_\oplus < e^{\beta((\Gamma_\oplus + \gamma_0)/3+\delta)}) \xrightarrow{\beta\to\infty} 1$$

$$2.2)\quad \mathbb{P}(e^{\beta((\Gamma_\oplus + \gamma_0)/3-\delta)} < \tau_+ < e^{\beta((\Gamma_\oplus + \gamma_0)/3+\delta)}) \xrightarrow{\beta\to\infty} 1$$

$$2.3)\quad \mathbb{P}\left(\frac{\tau_{\oplus +}}{\tau_\oplus} < e^{\beta((\Gamma_\oplus - \Gamma_+)/3-\delta)}\right) \xrightarrow{\beta\to\infty} 1$$

Note that points 1.1), 2.1) and 2.1), 2.2) in Theorem 1 imply the existence of a sort of dynamical phase transition: when crossing the $\lambda = 0$ line, the ratio between τ_+ and $\tau_\oplus$ passes from being "approximately 1" (for $\lambda > 0$) to being very large ($\lambda < 0$). Points 1.3) and 2.3) give a much stronger result describing the time interval elapsing from the arrival of the non-minuses to the arrival of the pluses. The length of this time interval is strongly related to the particular configuration at the hitting time to $\oplus$; since we do not have a detailed control of this configuration, we can only estimate this time from above.

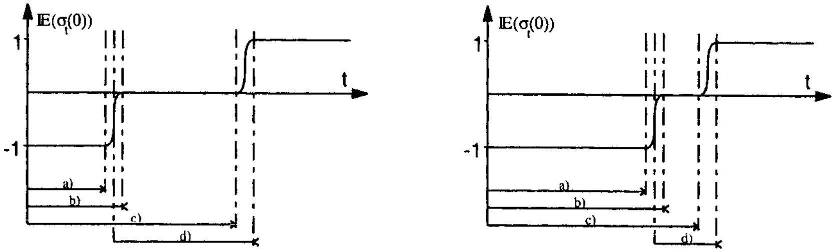


Fig. 2. Typical relaxation times described in Proposition 1.

Theorem 1 is a direct consequence of the following Proposition, concerning asymptotic behavior of $\tau_{\oplus}$, τ_{+}, $\tau_{\oplus +}$, expressed in terms of k_0, k_+, k_* and valid in the whole region of parameters $h > |\lambda|$.

Proposition 3.1. $\exists h_0$ such that if $h_0 > h > |\lambda| > 0$, then $\forall\, \delta > 0$:

$$a)\quad \mathbb{P}(\tau_{\oplus} > e^{\beta(k_0 - \delta)}) \xrightarrow{\beta \to \infty} 1 \tag{3.1}$$

$$b)\quad \mathbb{P}(\tau_{\oplus} < e^{\beta(k_0 + \delta)}) \xrightarrow{\beta \to \infty} 1 \tag{3.2}$$

$$c)\quad \mathbb{P}(\tau_{+} > e^{\beta(k_+ - \delta)}) \xrightarrow{\beta \to \infty} 1 \tag{3.3}$$

$$d)\quad \mathbb{P}(\tau_{\oplus +} < e^{\beta(k_* + \delta)}) \xrightarrow{\beta \to \infty} 1 \tag{3.4}$$

Proof of Theorem 1. Clearly, from (3.2) and (3.4), it easily follows that $\forall\, \delta > 0$

$$\mathbb{P}(\tau_{+} < e^{\beta(\max\{k_0, k_*\} + \delta)}) \xrightarrow{\beta \to \infty} 1 \tag{3.5}$$

and from (3.2) and (3.3) it easily follows that for $k_+ > k_0$,

$$\mathbb{P}(\tau_{\oplus +} > e^{\beta(k_+ - \delta)}) \xrightarrow{\beta \to \infty} 1 \tag{3.6}$$

Then, 1.1) and 2.1) of Theorem 1 directly follow from (3.1) and (3.2), 1.2) follows from (3.3) and (3.5), 2.2) follows from (3.1) and (3.5), and finally 1.3) follows from (3.2) and (3.6) while 2.3) from (3.1) and (3.4) (see (2.22), (2.23), (2.24) for the definitions of k_0, k_+ and k_*). ∎

We will prove Proposition 3.1 a), b), c), d) in Sections 6, 7, 8 and 9.

Theorem 2. Let $h > \lambda > 0$.

Let η^0 be the configuration where Q is full of zeroes and $\Lambda^0 \setminus Q$ is full of minuses.

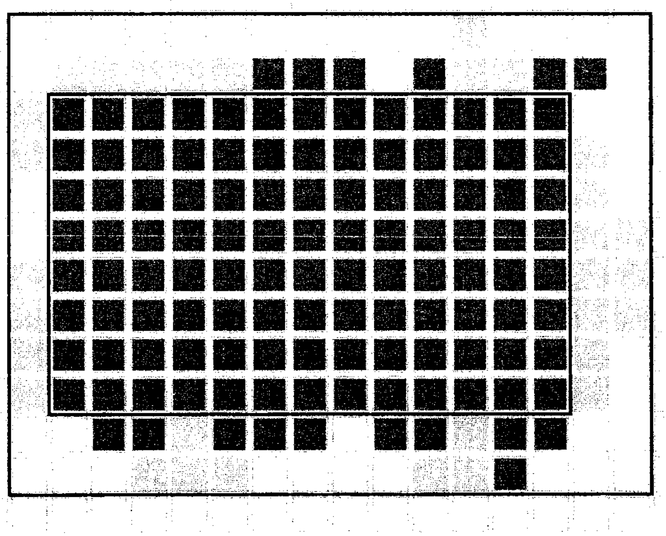


Fig. 3. Typical shape of large droplets described in Theorem 2.

Let us consider the process $\sigma^{\eta^0_0}_{\Lambda^0;\,t}$. Let $R^{\oplus}_{ext}(t)$ be the smallest rectangle containing the section at time t of $\mathscr{C}^{\oplus}_{0,0}(\sigma^{\eta^0_0}_{\Lambda^0;\,t}, t)$.

Let us consider a time T such that $\frac{1}{\beta}\ln T \in\,]k_*, k_0[$.

Then, with probability tending to one as $\beta \to \infty$, at time T there is a rectangle $R^+_{int}(T)$ made of $+$infected sites inside $R^{\oplus}_{ext}(T)$ such that $\forall\, r < \hat{r} := \min\{\frac{\gamma_0}{2}, k - k_+ - \frac{\gamma_0}{2}\}$,

$$\frac{\operatorname{diam}(R^{\oplus}_{ext}(T))}{\operatorname{diam}(R^{+}_{int}(T))} \leqslant 1 + e^{-\beta r} \tag{3.7}$$

This Theorem will be proved in Section 10.

4. THE EXIT PROBLEM

The problem of metastability in the Freidlin–Wentzell regime (namely, finite volume and $\beta \to \infty$) can be successfully studied in the framework of the exit problem theory for Markov chains (see [OS], [CaCe]).

We will see in Lemma 6.5 that the formation of the critical droplet in infinite volume is still a local phenomenon (it only depends on the sites near the droplet) and can be studied in a suitable finite volume. This is basically due to the fact that our dynamics is single-spin-flip and the flip rates have a local dependence on the configuration.

The main new issue is that now we must study the finite-volume exit problem at times much smaller than the typical exit time. Heuristically, since the volume is very large, supercritical droplets appear very rapidly. Hence, our need to describe the exit problem at small times.

This problem was solved in [DSch2] in the particular case of the kinetic Ising model but it was not studied in the classical approaches to metastability. In this Section, we use the methods developed in [OS] to solve the problem of "fast nucleation" in a general setting.

Let us give some definitions concerning a general Metropolis Markov chain in the so called Freidlin–Wentzell regime: Let $\mathscr{X}$ be a finite configuration space. Consider an ergodic, aperiodic Markov chain with transition probability from the configuration η to the configuration $\eta' \neq \eta$ given by

$$P(\eta, \eta') = q(\eta, \eta') \exp(-\beta[H(\eta') - H(\eta)]^+)$$

where $H: \mathscr{X} \to \mathbb{R}$ is called energy function and $q(\eta, \eta') = q(\eta', \eta)$ is an irreducible Markov kernel.

We define the *first hitting time* to the set $\mathscr{Q} \subset \mathscr{X}$ as:

$$\tau_{\mathscr{Q}} := \min\{t > 0 : \sigma_t \in \mathscr{Q}\}$$

η, η' are called *communicating* configurations if $P(\eta, \eta') > 0$. A *path* ω is a sequence $\omega := \{\eta_1, ..., \eta_n\}$, $n \in \mathbb{N}$, where η_j, η_{j+1}, $j = 1, ..., n-1$ are communicating configurations. We write $\omega: \eta \to \eta'$ to denote a path starting from η and ending in η'.

A set $\mathscr{Q} \subset \mathscr{X}$ is called *connected* if $\forall\, \eta$, $\eta' \in \mathscr{Q}$ there exists a path $\omega: \eta \to \eta'$ entirely contained in $\mathscr{Q}$.

We say that a configuration η is *downhill connected* to a configuration η' if there exists a path $\omega = (\eta_0 = \eta, \eta_1, ..., \eta_k = \eta')$ with $H(\eta_{i+1}) \leqslant H(\eta_i)$, $i = 0, ..., k-1$. We write $\omega: \eta \searrow \eta'$ to denote such a path.

We write $F(\mathscr{Q}) := \{\sigma \in \mathscr{Q} : H(\sigma) = \min_{\sigma'' \in \mathscr{Q}} H(\sigma'')\}$ and $U(\mathscr{Q}) := F(\partial^+ \mathscr{Q})$.

By abuse of notation, we write $H(\mathscr{Q})$ to mean the energy of any configuration in a set $\mathscr{Q}$ with $H(\eta) = const.\ \forall \eta \in \mathscr{Q}$.

η is called *stable configuration* if $H(U(\{\eta\})) > H(\eta)$, namely, if η is a local minimum of the energy. We will denote by $\mathscr{M}$ the set of stable configurations.

We call *depth* $\Gamma(\mathscr{Q}) := H(U(\mathscr{Q})) - H(F(\mathscr{Q}))$ the energy gap between $F(\mathscr{Q})$ and $U(\mathscr{Q})$.

We call *cycle* a connected set A such that $H(U(A)) > \max_{\sigma \in A} H(\sigma)$.

A cycle A for which there exists $\eta^* \in U(A)$ downhill connected to a point η in A^c with $H(\eta) < H(U(A))$, is called *transient*; given a transient cycle A the points $\eta^* \in U(A)$ downhill connected to $\eta \in A^c$ with $H(\eta) < H(U(A))$ are called *minimal saddles*. The set of the minimal saddles of a transient cycle A is denoted by $\mathscr{S}(A)$. More generally, given a set of configurations $\mathscr{Q}$, we denote by

$$\mathscr{S}(\mathscr{Q}) \tag{4.1}$$

the (possibly empty) set of configurations $\eta \in U(\mathscr{Q})$ which are downhill connected to some $\bar{\eta} \in \mathscr{Q}^c$ with $H(\bar{\eta}) < H(U(\mathscr{Q}))$.

A transient cycle A such that $\exists \bar{\eta} \notin A$ with $H(\bar{\eta}) < H(F(A))$, $\exists \eta^* \in \mathscr{S}(A)$ and a path $\omega: \eta^* \to \bar{\eta}$ below η^* (namely $\forall \eta \in \omega$, $\eta \neq \eta^*$, $H(\eta) < H(\eta)$), is called *metastable*.

Given a stable state η, we consider the smallest transient cycle A containing it; we call A *strict basin of attraction* of η and denote it by

$$\bar{B}(\eta) \tag{4.2}$$

For each pair of configurations $\eta, \eta' \in \mathscr{X}$ we define the set of their *minimal saddles* $\mathscr{S}(\eta, \eta')$ as follows:

Let for any path ω

$$\hat{H}(\omega) := \max_{\eta \in \omega} H(\eta)$$

The *communication height* between η and η' is

$$\bar{H}(\eta, \eta') := \min_{\omega: \eta \to \eta'} \hat{H}(\omega)$$

We set:

$$\mathscr{S}(\eta, \eta') = \{\zeta : H(\zeta) = \bar{H}(\eta, \eta'); \exists \omega: \eta \to \eta', \omega \notin \zeta : \max_{\xi \in \omega} H(\xi) = \bar{H}(\eta, \eta')\} \tag{4.3}$$

We define the saddle between sets of configurations in the natural way: $\mathscr{S}(\mathscr{Q}, \mathscr{Q}') := F(\{\bigcup_{\eta \in \mathscr{Q}, \eta' \in \mathscr{Q}'} \mathscr{S}(\eta, \eta')\})$.

The saddles between stable configurations ($\in \mathscr{M}$) will be called *natural saddles*.

We call *largest inner resistance* $\Theta(\mathscr{Q})$ of the set $\mathscr{Q}$ the maximum depth of any cycle contained in $\mathscr{Q}$ that does not contain the whole $F(\mathscr{Q})$:

$$\Theta(\mathscr{Q}) := \max_{\substack{A' \subset \mathscr{Q} \text{ s.t.} \\ F(\mathscr{Q}) \not\subset A'}} \Gamma(A') \tag{4.4}$$

If such a cycle does not exist, we set $\Theta(\mathscr{Q}) = 0$.

We denote by $C(A)$ the set of natural saddles inside the cycle A and by $\bar{C}(A) \subset C(A)$ the set of natural saddles with maximal energy.

A key property of cycles (Proposition 3.6 in [OS]) is that every cycle A such that $C(A)$ is non-empty can be uniquely decomposed as:

$$A = \tilde{A} \cup \bar{C}(A) \cup V \tag{4.5}$$

where $\tilde{A}$ is a collection of cycles A_i such that $H(U(A_i)) \equiv H(\bar{C}(A))$ and V is such that for every configuration $\eta \in V$ there exists $\omega: \eta \searrow \tilde{A}$. Moreover, $\tilde{A} \cup \bar{C}(A)$ is a connected set. Notice that $F(A) \subset \tilde{A}$. We generalize this decomposition to the case $C(A) = \varnothing$ by setting $\tilde{A} := F(A)$ and $V := A \setminus F(A)$.

Lemma 4.1. $\forall\, \delta > 0$, $\forall \kappa < \Gamma(A)$ and sufficiently large β,

$$\sup_{\eta \in F(A)} \mathbb{P}_\eta(\tau_{\partial^+ A} \leqslant e^{\beta\kappa}) \leqslant e^{-\beta(\Gamma(A) - \kappa - \delta)} \tag{4.6}$$

Proof. The proof is a straightforward consequence of the reversibility of Metropolis dynamics. Indeed, $\forall \eta, \xi \in \mathscr{X}$ such that $H(\xi) > H(\eta)$ we have:

$$\mathbb{P}_\eta(\tau_\xi < T) \leqslant e^{-\beta(H(\xi) - H(\eta))} T \tag{4.7}$$

(see the proof of Lemma 1 in [KO]). From (4.7) the thesis immediately follows. ∎

Let us now recall two of the main results in [OS]:

Proposition 4.2. Given a cycle A, $\forall k > \Gamma(A)$ $\forall \xi \in A$

$$\mathbb{P}_\xi(\tau_{\partial A} < e^{\beta k}) \xrightarrow{\beta \to \infty} 1 \tag{4.8}$$

$\forall \eta \in \partial A$, $\forall \varepsilon > 0$ and for β sufficiently large

$$\mathbb{P}_\xi(X_{\tau_{\partial A}} = \eta) \geqslant e^{-\beta(H(\eta) - H(u(A)) + \varepsilon)} \tag{4.9}$$

Proof. For the proof, see Proposition 3.7 in [OS]. ∎

The following Lemma is the counterpart of Lemma 4.1 and extends (4.8), (4.9).

Lemma 4.3. Given a non-trivial cycle A and a positive number k:

$$\Theta(A) < k \leqslant \Gamma(A)$$

we have $\forall \xi \in A$, $\forall \eta \in \partial A$, $\forall \varepsilon > 0$ and β sufficiently large

$$\mathbb{P}_\xi(\tau_{\partial A} < e^{\beta k}, X_{\tau_{\partial A}} = \eta) \geqslant e^{-\beta(H(\eta) - H(F(A)) - k + \varepsilon)} \tag{4.10}$$

Proof. The proof is very similar to the one of Proposition 3.7 in [OS]; again, we use induction on the number of internal saddles.

We suppose $|C(A)| = n+1$ and assume (4.10) to be valid for every cycle A' with $|C(A)| \leqslant n$.

We use the decomposition (4.5) and discern the subcycles in $\tilde{A}$ that intersect $F(A): \tilde{A} = \bigcup_{i=1}^h A_i \bigcup_{i=1}^j A_{h+i}$, $h \geqslant 1$, $j \geqslant 0$, $A_i \cap F(A) \neq \varnothing$ for $i = 1, ..., h$, $A_i \cap F(A) = \varnothing$ for $i = h+1, ..., h+j$, $U(A_i) \cap \bar{C}(A) \neq \varnothing$ for $i = 1, ..., h+j$.

We distinguish two cases (see Fig. 4):

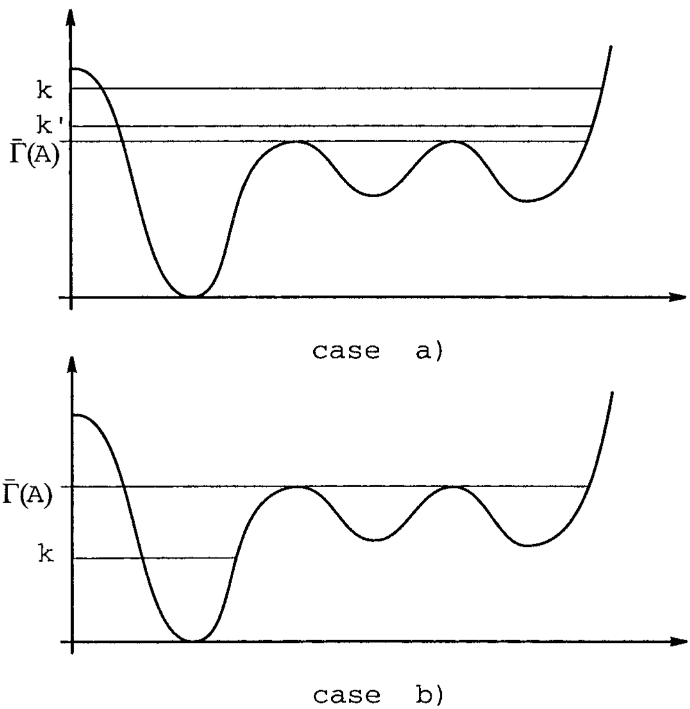


Fig. 4. Relevant quantities in a one-dimensional case.

a) $k > H(\bar{C}(A)) - H(F(A)) \equiv \bar{\Gamma}(A)$, where we use (4.8), (4.9)

b) $k \leqslant \bar{\Gamma}(A)$, where we use the inductive hypothesis on the cycles in $\tilde{A}$ which is certainly valid since, of course, $|C(A_i)| \leqslant n$, $i = 1, ..., h+j$.

Notice that (since we only consider $k > \Theta(A)$) when $h(A) \geqslant 2$ (i.e. when in $\tilde{A}$ there are at least two cycles with non-empty intersection with $F(A)$) we necessarily have $\Theta(A) = \Gamma(A_1) = \bar{\Gamma}(A)$ and only case a) is possible. It will be clear in the proof of case b) that if we had considered the case $k \leqslant \Theta(A)$ we could not have handled it with our construction. This is not a limit of our proof but a crucial feature of the exit problem at very small times. When considering such small times, the inner structure of the cycles becomes relevant and counterexamples where (4.10) is not valid are easy to find.

Case a): Let us first prove

$$\mathbb{P}_\xi(\tau_{\partial A} < e^{\beta k}) \geqslant e^{-\beta(\Gamma(A) - k + \varepsilon)} \tag{4.11}$$

$\forall \varepsilon > 0$, β sufficiently large.

Given $k' \in]\bar{\Gamma}(A), k[$, let us divide the time interval $e^{\beta k}$ into $\lfloor e^{\beta k}/(\lfloor e^{\beta k'} \rfloor + 1) \rfloor$ subintervals larger than $e^{\beta k'}$.

We take $k' = \bar{\Gamma}(A) + \varepsilon'$ with a suitable choice of ε'.

We have, by using Markov property

$$\mathbb{P}_\xi(\tau_{\partial A} > e^{\beta k}) \leqslant [\sup_{\zeta \in A} \mathbb{P}_\zeta(\tau_{\partial A} > e^{\beta k'})]^{e^{\beta(k-k')}} \tag{4.12}$$

Let $\xi \in A$, $T_1 = e^{\beta k'}$. We define an event $\mathscr{E}_{\xi, T_1}$ analogous to the one introduced in the proof of Proposition 3.7 in [OS]. $\mathscr{E}_{\xi, T_1}$ consists in a set of trajectories starting from ξ and exiting A within the time T_1 in a particular manner:

Given η^* in $U(A)$ there is a downhill path $\bar{\xi}_0, \bar{\xi}_1, ..., \bar{\xi}_{m-1}, \bar{\xi}_m$ with $\bar{\xi}_0 = \eta^*$, $\bar{\xi}_1, ..., \bar{\xi}_{m-2} \in V$, $\bar{\xi}_{m-1} \in \partial\tilde{A}$, $\bar{\xi}_m \in \tilde{A}$. It may happen that $\eta^* \in \partial\tilde{A}$. In that case, we set $k = 1$ and the path is just one step. We call A_{j^*} the cycle in $\tilde{A}$ to which $\bar{\xi}_m$, belongs.

The trajectories in $\mathscr{E}_{\xi, T_1}$ first enter $\tilde{A}$ within time $T_1/4$ then, by following a suitable finite sequence of cycles in $\tilde{A}$ they hit A_{j^*} in a time shorter than $T_1/4$; subsequently, they exit from A_{j^*}. within another interval $T_1/4$ through $\bar{\xi}_{m-1}$. Finally, they follow without hesitation the uphill path $\bar{\xi}_{m-1}, \bar{\xi}_{m-2}, ..., \bar{\xi}_1, \eta$; obviously, for β large enough this last segment takes place within $T_1/4$. We refer to the proof of Proposition 3.7 in [OS] for more details.

We have by (4.8), (4.9) applied to the sequence of cycles in $\tilde{A}$, since $k' > \Gamma(A_l)$ $l = 1, ..., h+j$

$$\mathbb{P}(\mathscr{E}_{\xi, T_1}) \geqslant e^{-\beta(H(U(A)) - H(\bar{C}(A)) + \varepsilon'')} \tag{4.13}$$

$\forall \varepsilon'' > 0$, β sufficiently large.

From (4.12) and (4.13), we have for β sufficiently large

$$\begin{aligned}\mathbb{P}_\xi(\tau_{\partial A} > e^{\beta k}) &\leqslant [1 - e^{-\beta(H(U(A)) - H(\bar{C}(A)) + \varepsilon'')}]^{e^{\beta(k - \bar{\Gamma}(A) - \varepsilon')}}\\ &\leqslant 1 - e^{-\beta(\Gamma(A) - k + \varepsilon)}\end{aligned}$$

$\forall \varepsilon > \varepsilon' + \varepsilon''$ and (4.11) follows.

To conclude the proof, we need some more definitions.

Given $\xi \in A$ we set

$$\begin{aligned}\theta_\xi &= \sup\{t < \tau_{\partial A} : X_t = \xi\}\\ \tau'_\xi &= \tau_{\partial A} - \theta_\xi\end{aligned} \tag{4.14}$$

(we set $\theta_\xi = \tau_{\partial A}$; $\tau'_\xi = 0$ when $\tau_\xi > \tau_{\partial A}$).

It follows from (4.8), (4.9) that $\forall \varepsilon > 0$ $\exists c > 0$:

$$\sup_{\xi \in F(A)} \mathbb{P}_\xi(\tau_\xi > e^{\beta(\Theta(A) + \varepsilon)}, \tau_{\partial A} > e^{\beta(\Theta(A) + \varepsilon)}) \leqslant e^{-e^{c\beta}} \tag{4.15}$$

for β sufficiently large.

To simplify notation we write SES to denote a superexponentially (in β) small quantity like r.h.s. of (4.15).

It follows from (4.15) that $\forall \xi \in F(A)$

$$\mathbb{P}_\xi(\tau'_\xi > e^{\beta(\Theta(A) + \varepsilon)}) \leqslant SES.$$

We have from (4.14), (4.15), taking into account that $k > \bar{\Gamma}(A)$

$$\begin{aligned}\mathbb{P}_\xi(\tau_{\partial A} \leqslant e^{\beta k}, X_{\tau_{\partial A}} = \eta) &\geqslant \mathbb{P}_\xi\left(\theta_\xi \leqslant \frac{e^{\beta k}}{2}, \tau'_\xi \leqslant \frac{e^{\beta k}}{2}, X_{\tau_{\partial A}} = \eta\right)\\ &= \mathbb{P}_\xi\left(\theta_\xi \leqslant \frac{e^{\beta k}}{2}, X_{\tau_{\partial A}} = \eta\right) - \mathbb{P}_\xi\left(\theta_\xi \leqslant \frac{e^{\beta k}}{2}, \tau'_\xi > \frac{e^{\beta k}}{2}, X_{\tau_{\partial A}} = \eta\right)\\ &\geqslant \mathbb{P}_\xi\left(\theta_\xi \leqslant \frac{e^{\beta k}}{2}, X_{\tau_{\partial A}} = \eta\right) - SES\end{aligned} \tag{4.16}$$

Now, it is easy to see that for any $T>0$, $\forall \xi \in A$

$$\mathbb{P}_\xi(\theta_\xi \leqslant T, X_{\tau_{\partial A}} = \eta) = \mathbb{P}_\xi(\theta_\xi \leqslant T)\ \mathbb{P}_\xi(X_{\tau_{\partial A}} = \eta) \tag{4.17}$$

Indeed,

$$\mathbb{P}_\xi(\theta_\xi \leqslant T, X_{\tau_{\partial A}} = \eta) = \left[1 + P(\xi,\xi) + \sum_{s=2}^{T-1} \sum_{\xi_1,\ldots,\xi_{s-1}\in A} P(\xi,\xi_1,\ldots,\xi_{s-1},\xi)\right] \times \left[P(\xi,\eta) + \sum_{l=2}^{\infty} \sum_{\eta_1,\ldots,\eta_{l-1}\in A\setminus(\xi)} P(\xi,\eta_1,\ldots,\eta_{l-1},\eta)\right] \tag{4.18}$$

On the other hand we have

$$\mathbb{P}_\xi(\theta_\xi \leqslant T) = \left[1 + P(\xi,\xi) + \sum_{s=2}^{T-1} \sum_{\xi_1,\ldots,\xi_{s-1}\in A} P(\xi,\xi_1,\ldots,\xi_{s-1},\xi)\right] \times \left[\sum_{\zeta\in\partial A} P(\xi,\zeta) + \sum_{l=2}^{\infty} \sum_{\eta_1,\ldots,\eta_{l-1}\in A\setminus\{\xi\}} P(\xi,\eta_1,\ldots,\eta_{l-1},\zeta)\right] \tag{4.19}$$

One has:

$$\mathbb{P}_\xi(\theta_\xi \leqslant \infty) = 1 \tag{4.20}$$

as it follows from an immediate estimate based on ergodicity of our chain with finite state space and Borel–Cantelli's Lemma.

From (4.18), (4.19), passing to the limit $T \to \infty$, we get:

$$\frac{P(\xi,\eta) + \sum_{l=1}^{\infty} \sum_{\eta_1,\ldots,\eta_{l-1}\in A\setminus\{\xi\}} P(\xi,\eta_1,\ldots,\eta_{l-1},\eta)}{\sum_{\zeta\in\partial A} P(\xi,\zeta) + \sum_{l=1}^{\infty} \sum_{\eta_1,\ldots,\eta_{l-1}\in A\setminus\{\xi\}} P(\xi,\eta_1,\ldots,\eta_{l-1},\zeta)} = \mathbb{P}_\xi(X_{\tau_{\partial A}} = \eta) \tag{4.21}$$

From (4.21), (4.18), (4.19), we conclude the proof of (4.17).

From (4.16), (4.17), we get taking $\xi \in F(A)$, $\zeta \in A$

$$\begin{aligned}\mathbb{P}_\zeta(\tau_{\partial A} \leqslant e^{\beta k}, X_{\tau_{\partial A}} = \eta) &\geqslant \mathbb{P}_\zeta\left(\theta_\xi \leqslant \frac{e^{\beta k}}{2}\right) \mathbb{P}_\xi(X_{\tau_{\partial A}} = \eta) - SES \\ &\geqslant \mathbb{P}_\zeta\left(\tau_{\partial A} \leqslant \frac{e^{\beta k}}{2}\right) \mathbb{P}_\xi(X_{\tau_{\partial A}} = \eta) - SES\end{aligned} \tag{4.22}$$

From (4.22) and (4.9), we conclude the proof in case a).

Let us now go to case b). Since $k > \Theta(A) \geqslant \Theta(A_1)$ and $k \leqslant \Gamma(A_1) = \bar{\Gamma}(A)$, we know that $h=1$ namely, in $\tilde{A}$ there is a unique cycle A_1 containing the whole $F(A)$.

We use the iterative hypothesis on A_1 (obviously, $|C(A_1)| \leqslant n$), while we use (4.8), (4.9) to handle the exit from the other subcycles since $k > \Gamma(A_j)\ \forall j \geqslant 2$.

Let $\bar{\xi}_1, \ldots, \bar{\xi}_m$ and A_{j^*}. be defined like in case a).

For any $\zeta \in A$ we consider again an event $\mathscr{E}_{\zeta, T_1}$ with $T_1 = e^{\beta k}$: The trajectories in $\mathscr{E}_{\zeta, T_1}$ first go to A_1 within a time $T_1/4$, following a suitable finite sequence of distinct cycles in $\tilde{A} \setminus A_1$ (by always exiting through $\bar{C}(A)$). Then they exit A_1 and follow an analogous sequence of distinct cycles in $\tilde{A} \setminus A_1$ up to A_{j^*} within another $T_1/4$. Finally they exit from a suitable $\bar{\xi}_{m-1}$ within another $T_1/4$ and go uphill up to $\eta^* \in \partial A$ like before.

The rest of the estimate goes as in the previous case.

So we get

$$\mathbb{P}(\mathscr{E}_{\xi, T_1}) \geqslant e^{-\beta(H(U(A_1)) - H(F(A)) - k + \varepsilon)} e^{-\beta(H(U(A)) - H(\bar{C}(A)) + \varepsilon)} \tag{4.23}$$

where we used that the exit from A_{j^*} followed by the fast ascent to η^* gives a contribution

$$e^{-\beta(H(U(A)) - H(\bar{C}(A)) + \varepsilon)}$$

From (4.23) we conclude the proof of

$$\mathbb{P}_{\xi}(\tau_{\partial A} < e^{\beta k}) \geqslant e^{-\beta(\Gamma(A) - k + \varepsilon)}$$

and then, with the previous argument, the proof of (4.10).

Notice that in present case b) we do not use recurrence and Markov property as in (4.12); indeed we could have obtained directly the result by simply substituting in our construction η to $\eta^* \in \partial A$ (not necessarily belonging to $U(A)$) by avoiding in this way the use of (4.17).

To conclude the proof of Lemma 4.3 we only need to prove the basis of the induction namely, the case of A completely attracted by a unique stable plateau $F(A)$.

In this case again we construct an event taking place in a finite (independent of β), sufficiently large interval T_0 consisting in descending from a generic $\zeta \in A$ to $F(A)$ and then ascending to $U(A)$ following an uphill path towards η.

The proof goes like that of case a). ∎

5. PRELIMINARY ANALYSIS OF THE DYNAMICS

In this section we give some more definitions and a brief survey of the growth and contraction mechanisms that will be useful in the following sections.

5.1. The Bootstrap Procedure

Let $A \subset \mathbb{Z}^2$ be a finite set. Let $nn_A(x)$ be the number of nearest neighbors of the site x belonging to A.

We recursively take $A_i := A_{i-1} \cup \{x \in \partial^+ A_{i-1} : nn_{A_{i-1}}(x) \geqslant 2\}$, where $A_0 := A$. It is easy to see that in a finite number of steps this procedure converges to a final set that we denote by $\mathrm{B}(A)$. Indeed every A_i is contained into the "rectangular envelope" of A (namely the smallest rectangle containing A). Since the rectangular envelope of A is a finite set, the procedure stops in a finite number of iterations and $\mathrm{B}(A)$ is clearly contained in the rectangular envelope of A. We call *bootstrap* this procedure.

Notice that $\mathrm{B}(A)$ is either a single rectangle or a set of rectangles whose mutual distance $\|\cdot\|_1$ is at least 2.

We use the same terminology to describe the analogous configuration-valued procedure. E.g. let $I^+(\eta)$ be the set of pluses in the configuration η, we call bootstrap of the pluses in η, the configuration $\mathrm{B}^+(\eta)$ obtained from η by setting to plus all the spins of the sites in $\mathrm{B}(I^+(\eta))$.

We call *picture frame envelope* of a set of $\oplus$-clusters, the configuration having pluses inside the smallest rectangle containing the set, zeros on the outer boundary of such rectangle and minuses elsewhere.

Let $\mathscr{C}^{\oplus}_{x,0}(\sigma, 0)$ be a cluster of zero-pluses and η be its picture-frame envelope. We say that the space-time cluster *exits its picture-frame envelope at time t* iff $t = \inf\{s > 0; \exists y \in \mathscr{C}^{\oplus}_{x,0}(\sigma, s) \text{ s.t. } \sigma_s(y) > \eta(y)\}$.

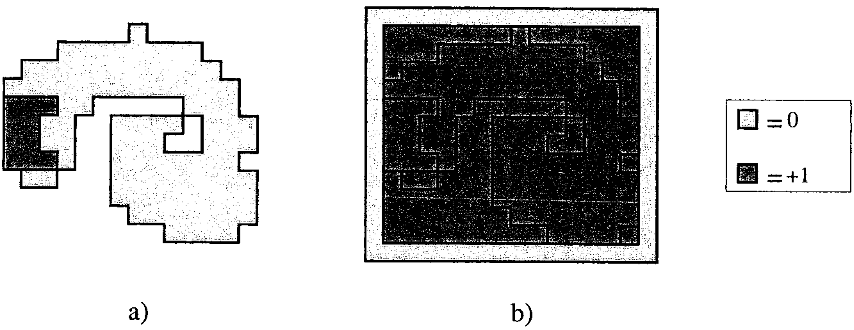


Fig. 5. Picture frame envelope.

The interesting feature of this configuration is that starting from a configuration containing a given ⊕-cluster, it is not possible to exit its picture-frame envelope without interacting with another droplet or overcoming a positive energy barrier.

5.2. Growth and Contraction Mechanisms

In order to understand the shape of the local minima of the hamiltonian, we briefly survey all possible downhill growth mechanisms: The region we are interested in is:

$$0 < |\lambda| < h \ll 1$$

i) growth of "zero-pluses" in the sea of minuses.

The formation energy of a zero from a minus is:

$$\Delta H = -\lambda' - h' = \lambda - h - 4 - 2(n^+ - n^-) \tag{5.1}$$

ΔH is negative iff $n^+ \geqslant n^- - 2$.

The only two cases with energy gain are a) $n^- \leqslant 2$ (i.e. two non-minus nearest neighbors), or b) $n^- = 3$ and $n^+ = 1$ (i.e. removal of a direct interface). These mechanisms are schematically displayed in Fig. 6.

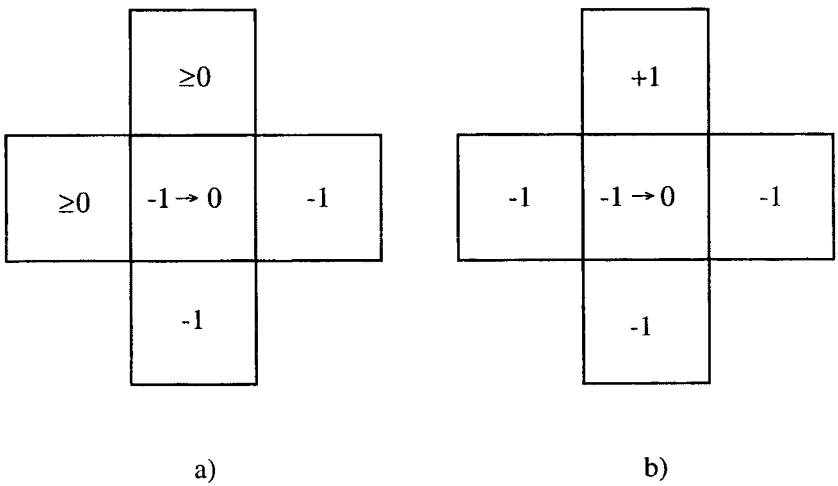


Fig. 6. Downhill growth mechanisms for the zero- pluses.

The first mechanism can be controlled with the bootstrap process: it cannot create a non-minus out of the region spanned by the bootstrap of the non-minuses.

The second mechanism can change to zero the minus sites directly neighboring a plus and hence it can at most add a layer of zeroes to a configuration.

The two mechanisms can neither create pluses out of the region spanned by the bootstrap of the initial configuration of non- minuses nor have zeroes out of the outer boundary of such region.

It is clear that in order for a $\oplus$-cluster to be able to exit its picture frame envelope, either there is another cluster closer than three units (in the $\|\cdot\|_1$ norm) from this region or a unit-square protuberance (of zeroes inside the minuses or of pluses inside the zeroes) must be formed.

ii) growth of minuses in the sea of non-minuses.

The energy needed to form a minus is:

$$\Delta H = +\lambda' + h' = -\lambda + h + 4 - 2(n^+ - n^-) \tag{5.2}$$

ΔH is negative if $n^+ = 0$ and $n^- \geqslant 3$

The droplets of minuses can always be eroded from the corners, though possible holes or inlets of non-minuses can be removed from inside. The region the minuses can span following a downhill path is contained inside the region spanned by the bootstrap of the non-minuses.

iii) growth of non-pluses in the sea of pluses.

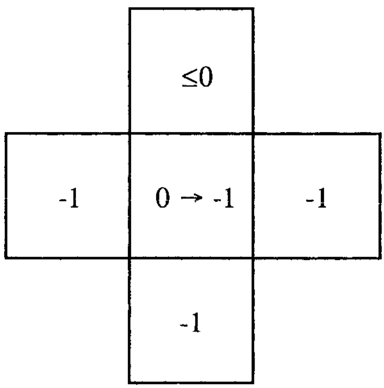


Fig. 7. Downhill transitions from zero to minus.

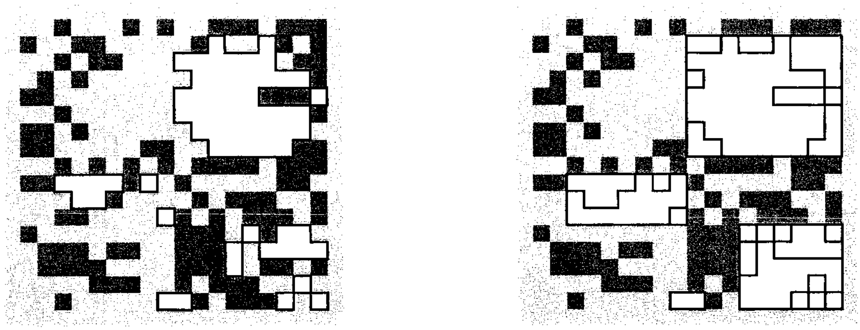


Fig. 8. Region that the minuses can span following a downhill path.

The energy needed to change a plus into a zero is:

$$\Delta H = -\lambda' - h' = -4 - \lambda + h + 2(n^+ - n^-) \tag{5.3}$$

ΔH is negative if $n^+ - n^- < 2$.

The cases with energy gain are $\{n^+ = 2, n^- \geqslant 1\}$ and $n^+ \leqslant 1$. The only configurations where a plus with two plus neighbors can change to zero lowering the energy contain a direct interface minus-plus.

Only inlets or isolated droplets can be removed from inside the non-pluses.

The droplets of non-pluses can always be eroded from the corners following a downhill path, being the transition $-1 \to 0 \to +1$ downhill if $n^+ \geqslant 2$.

6. THE FIRST METASTABLE REGIME: PROOF OF PROPOSITION 3.1 CASE a)

6.1. Finite-Volume Energy Landscape

We start our analysis by re-deriving and extending to our new dynamics the finite-volume results obtained in [CiO]. It is possible to show (see [M]) that for $\lambda < 4$ the sets of local minima are the same in the two dynamics and moreover, that the saddle between any couple of local minima coincide in the two dynamics. Here, we directly study the new dynamics in the whole region $0 < |\lambda| < h \ll 1$. Indeed, we are particularly interested in the region near the $\lambda = 0$ line, which was not explicitly studied in [CiO].

Let us introduce some more definitions that will be used in this Section:

Consider the process $\bar{\sigma}_{Q;t}$ in the finite volume Q under periodic boundary conditions and the associated hamiltonian (see (2.12)). For the process $\bar{\sigma}_{Q;t}$, we call *maximal subcritical cycle*

$$\mathscr{A} \tag{6.1}$$

the largest (in the sense of inclusion) metastable cycle containing $-\underline{1}$ and not intersecting $\oplus$. An equivalent definition of $\mathscr{A}$ will turn out to be "the set of all configurations having the saddle with $\oplus$ above the saddle with $-\underline{1}$" (indeed this set is connected). Here, with "subcritical" we mean a configuration that, with probability tending to one, evolves towards $-\underline{1}$ before $+\underline{1}$.

We will see (Lemma 6.3) that in a sufficiently large square Q, $\mathscr{A} = \mathscr{A}(Q)$ is the cycle (w.r.t. the above dynamics) given by the maximal connected component containing $-\underline{1}$ with depth lower than $\Gamma_{\oplus}$.

We will make use of the process

$$\tilde{\sigma}^{\eta}_{\Lambda;t} := \tilde{\sigma}^{\mathscr{A};\eta}_{\Lambda;t} \tag{6.2}$$

restricted to the maximal subcritical cycle $\mathscr{A}$ (see (2.15)).

We will largely use the methods introduced in [CiO] to determine supercriticality or subcriticality of local minima, focusing on the problem of finding the largest inner resistance of the maximal subcritical cycle $\mathscr{A}$.

We call *cost* of the path ω the quantity

$$\sup_i \sup_{j \geqslant i} [H(\omega_j) - H(\omega_i)]^+ \tag{6.3}$$

We call *wrapping* a configuration with a cluster of zero- pluses winding around the torus Q. It will be clear in the following analysis that for large enough D, wrapping configurations are outside the maximal subcritical cycle $\mathscr{A}$.

As it has been shown in [CiO], the non-wrapping minima of the energy are families of *plurirectangles*, namely configurations containing rectangles of zeroes with sidelengths greater than 1 and at least two units away from each other in the $\|\cdot\|_1$ distance. *In the interior* of the rectangles of zeroes, these configurations can have rectangles of pluses with sides larger than 1 and at least two units away from each other in the $\|\cdot\|_1$ distance. Here and in what follows *"in the interior"* means "inside and at least one unit away from the boundary" of the rectangles of zeroes.

We call *birectangle* $R(L^0, l^0, L^+, l^+)$ (where $L^0 \geqslant l^0 \geqslant 2$ and either $L^+ \geqslant l^+ \geqslant 2$ or $L^+ = l^+ = 0$) the set of configurations containing a single rectangle $L^0 \times l^0$ of zeroes with a single rectangle $L^+ \times l^+$ of pluses in its interior. Clearly $L^0 \geqslant L^+ + 2$, $l^0 \geqslant l^+ + 2$.

Heuristically, we consider three possibilities: both the rectangles of zeroes and the rectangles of pluses inside them tend to shrink; some rectangle of zeros tend to grow; some rectangle of pluses tend to grow while all rectangles of zeroes tend to shrink. The first and the second case clearly correspond to subcritical and supercritical configuration, respectively, while the third case is more complex and deserves further investigation.

We call picture frame $F(L^+, l^+) := R(L^+ + 2, l^+ + 2, L^+, l^+)$ the set of all configurations consisting of a single rectangle $L^+ \times l^+$ (where $L^+ \geqslant l^+ \geqslant 2$) of pluses surrounded by a unitary layer of zeroes. We consider $-\underline{1}$ and wrapping minima as degenerate cases of plurirectangles.

As it has been shown in [CiO], the analysis of the criticality of the rectangles traces back to the comparison between the energy of a unit-square protuberance (namely γ_0 or γ_+ defined in (2.20) and (2.21)), and the energies

$$\varphi_0(\ell) := (\ell - 1)(h - \lambda) \tag{6.4}$$

or

$$\varphi_+(\ell) := (\ell - 1)(h + \lambda) \tag{6.5}$$

that correspond to the erosion of the first $\ell - 1$ sites of a rectangle with the smallest side-length ℓ, if the rectangle is made of zeroes inside the minuses or pluses inside the zeroes, respectively. Indeed, the depth of the strict basin of attraction of a rectangle of zeroes, with the smallest side-length ℓ, is either γ_0 or $\varphi_0(\ell)$ (see the proof of Lemma 6.1 below). The same argument applied to the rectangles of pluses leads to the comparison between γ_+ and $\varphi_+(\ell)$. This analysis leads to the introduction of the critical lengths in (2.17).

Let us consider a birectangle $R(L^0, l^0, L^+, l^+)$. The meaning of the critical lengths in (2.17) is the following:

$$\gamma_+ > \varphi_+(l^+) \Leftrightarrow l^+ < L^* \tag{6.6}$$

$$\gamma_+ > \varphi_0(l^0) \Leftrightarrow l^0 < \tilde{M} \tag{6.7}$$

$$\gamma_0 > \varphi_0(l^0) \Leftrightarrow l^0 < M^* \tag{6.8}$$

$$\gamma_0 > \varphi_+(l^+) \Leftrightarrow l^+ < \tilde{L} \tag{6.9}$$

$$\gamma_0 + \gamma_+ > \varphi_+(l^+) + \varphi_0(l^+ + 2) \Leftrightarrow l^+ < \ell^* \tag{6.10}$$

The cases $l^0 > M^*$ or $l^+ < L^*$ easily lead back to the Ising-like case $\mathcal{I}^-$ (see (2.16)). Indeed, we will show that a configuration η consisting of a single rectangle of zeroes is such that

$$\eta \in \mathcal{A} \Leftrightarrow l^0 < M^* \tag{6.11}$$

On the other hand, if η' is a local minimum with the smallest side of the largest rectangle of zeroes equal to l^0 and with the smallest side of the largest rectangle of pluses equal to l^+, then we have the following implication:

$$l^0 < M^* \qquad \text{and} \qquad l^+ < L^* \Rightarrow \eta' \in \mathcal{A} \tag{6.12}$$

The case $l^0 < M^*$ and $l^+ > L^*$ leads to a complex growth-and-contraction mechanism involving both the zeroes and the pluses. Indeed, the formation of a direct interface -1, $+1$ is highly unlikely, so that the relationships $L^0 \geqslant L^+ + 2$ and $l^0 \geqslant l^+ + 2$ are typically preserved during the evolution. Therefore, there is a competition between the shrinking of the rectangle of zeroes and the growth of the pluses inside it.

We use the constructive criterion introduced in [OS] to determine the energy of $\mathcal{S}(-\underline{1}, \oplus)$ and $\Theta(\mathcal{A})$: The criterion consists in finding a set $\mathcal{G}$ of configurations with the following properties:

1. $\mathcal{G}$ is connected and $\forall \bar{\eta} \in \mathcal{S}(\mathcal{G})$
2. $\exists \omega: \bar{\eta} \to -\underline{1}$ s.t. $\omega \setminus \bar{\eta} \subset \mathcal{G}$ and $H(\xi) < H(\bar{\eta})$ $\forall \xi \in \omega \setminus \bar{\eta}$
3. $\exists \omega': \bar{\eta} \to \oplus$ s.t. $\omega' \subset \mathcal{G}^c$ and $H(\xi') < H(\bar{\eta})$ $\forall \xi' \in \omega' \setminus \bar{\eta}$.

These properties ensure that $\mathcal{S}(\mathcal{G}) \equiv \mathcal{S}(-\underline{1}, \oplus) = \mathcal{S}(\mathcal{A})$ and that $\mathcal{A} \subset \mathcal{G}$ (see (4.1) for the definition of $\mathcal{S}$).

In the following, we define $\mathcal{G}$ on geometrical grounds and bound the inner resistance of $\mathcal{A}$ with the inner resistance of $\mathcal{G}$.

We write

$$m^*(L) := \tilde{L} + \frac{h - \lambda}{h + \lambda} (\tilde{M} - 2 - L) \tag{6.13}$$

and

$$f(L) := \min\{\ell^*, \max\{\tilde{L}, m^*(L)\}\} \tag{6.14}$$

Consider a non-wrapping minimum $\eta \in \mathcal{M}$ (we recall that $\mathcal{M}$ is the set of all local minima of the hamiltonian). Let $L_i^0(\eta)$, $l_i^0(\eta)$ with $L_i^0(\eta) \geqslant l_i^0(\eta) \geqslant 2$ be the side-lengths of the i-th rectangle of zeroes in η and let

$L^0(\eta) := \max_i L_i^0(\eta)$, $l^0(\eta) := \max_i l_i^0(\eta)$. Let $L_j^+(\eta)$, $l_j^+(\eta)$, $L^+(\eta)$, $l^+(\eta)$ be the analogous side-lengths for the rectangles of pluses. We extend in the natural way this definitions to wrapping minima by setting, when necessary, these lengths equal to D.

We define a map $\Psi\colon \mathscr{M} \to \mathscr{M}$ in the following steps:

i) We erase (by setting the spins to zero) all the rectangles of pluses with minimum side-length l_i^+ shorter than L^*.

ii) For every rectangle of zeroes, we consider all remaining rectangles of pluses inside it and take the smallest rectangle containing all of them. We fill this rectangle with pluses.

We denote by $\bar{\mathscr{M}}$ the set of local minima η such that:

$$l^0(\Psi(\eta)) < M^* \tag{6.15}$$

and, for every i, one of the following conditions is verified:

1. $l_i^+(\Psi(\eta)) = 0$
2. $l_i^+(\Psi(\eta)) \geqslant L^*$, $\max\{l^0(\Psi(\eta)), L_i^+(\Psi(\eta))+2\} < \tilde{M}$ and $l^+(\Psi(\eta)) < f(L_i^+(\Psi(\eta)))$
3. $l_i^+(\Psi(\eta)) \geqslant L^*$, $\max\{l^0(\Psi(\eta)), L_i^+(\Psi(\eta))+2\} \geqslant \tilde{M}$ and $l^0(\Psi(\eta)) < f(L_i^+(\Psi(\eta)))+2$

Notice that in the case $h > -\lambda > 0$, condition 1. is automatically verified for $l^0 < M$.

We denote by $\mathscr{G}$ the set of configurations η such that every downhill path starting from η ends in $\bar{\mathscr{M}}$ (basin of attraction of $\bar{\mathscr{M}}$).

We remark that the set $\tilde{\mathscr{G}} := \{\eta \text{ such that } \mathbb{P}_\eta(\tau_{-\underline{1}} < \tau_\oplus) \xrightarrow{\beta\to\infty} 1\}$ is in general larger than $\mathscr{G}$. We choose to restrict ourselves to $\mathscr{G}$ instead of analyzing $\tilde{\mathscr{G}}$ as $\Theta(\tilde{\mathscr{G}})$ can provide a too large upper estimate for $\Theta(\mathscr{A})$: indeed, for suitable values of h, λ we can have configurations $\eta \in \tilde{\mathscr{G}}$ with $\ell^* \leqslant l^+(\Psi(\eta)) < \tilde{L}$; for these configurations the minimum cost of a path $\omega\colon \eta \to -\underline{1}$ exceeds $\Theta(\mathscr{A})$ (notice that in the region studied in [CiO] this case cannot take place since $\tilde{L} < \ell^*$).

Lemma 6.1. 1. $\mathscr{G}$ is connected

2. $\Theta(\mathscr{G})\gamma_0$.

Proof. In order to prove the thesis, it is sufficient to prove that $\forall \eta \in \bar{\mathscr{M}}$ there exists a path, entirely contained in $\mathscr{G}$ and with a cost smaller than γ_0, that connects η to $-\underline{1}$.

In the construction of this path, we will associate to every local minimum $\eta \in \bar{\mathscr{M}}$ a suitable transient cycle that we will denote by $\mathscr{B}(\eta)$, such that $F(\mathscr{B}(\eta)) = \{\eta\}$. In most of the cases $\mathscr{B}$ will coincide with the strict basin of attraction of η (see (4.2)). The introduction of these cycles allows to focus the attention on the local minima in $\bar{\mathscr{M}}$ instead of dealing with the whole $\mathscr{G}$.

Though it is not necessary for the proof, we describe $\mathscr{S}(\mathscr{B}(\eta))$ for the various kinds of local minima of the hamiltonian; in particular, we will often consider possible "growing patterns" and compare the associated energy $H(\mathscr{S}(\mathscr{B}(\eta)))$. We will refer to these results in the proof of Lemma 6.3.

For every $\eta \in \bar{\mathscr{M}}$, we will construct a path $\omega: \eta \to -\underline{1}$, $\omega \in \mathscr{G}$, with a cost smaller or equal to

$$\bar{\Theta} := \max_{\xi \in \bar{\mathscr{M}}} \Gamma(\mathscr{B}(\xi))$$

depth of $\mathscr{B}(\xi)$ for $\xi \in \bar{\mathscr{M}}$. In this way, $\Theta(\mathscr{A})$ will be bounded by $\bar{\Theta}$. To find the path $\omega: \eta \to -\underline{1}$, we will observe that, with our definition of $\mathscr{B}$, $\forall \eta' \in \bar{\mathscr{M}}$, $\mathscr{S}(\mathscr{B}(\eta'))$ is completely attracted by the minima in $\bar{\mathscr{M}}$ (namely every downhill path starting from $\mathscr{S}(\mathscr{B}(\eta))$ ends in $\bar{\mathscr{M}}$) and find a sequence of minima $\{\eta_k\}_{k \leqslant K} \in \bar{\mathscr{M}}$ with $\mathscr{B}(\eta_0) \ni \eta$, $\eta_K = -\underline{1}$ such that $\mathscr{S}(\mathscr{B}(\eta_k))$ is downhill connected with η_{k+1}.

We call *relevant minimum* a configuration $\eta \in \bar{\mathscr{M}}$ consisting either of a rectangle of zeroes or of a picture-frame $F(L^+, l^+)$ with $l^+ \geqslant L^*$. It will turn out that the geometrical shapes of relevant minima are "self reproducing" along the typical trajectories of our dynamics; for instance a relevant picture-frame will evolve along a sequence of picture frames whereas an irrelevant picture frame will not. Since relevant minima will play a special role in the construction of the path ω, we will analyze the energy landscape around them.

We remark that our definitions and notation are conceived for the general case and in some region of the parameters there is no need for such a generality (e.g. if $\lambda < 0$ relevant picture-frames in $\mathscr{G}$ do not even exist).

We now give the definitions of $\mathscr{B}(\eta)$ and characterize $\mathscr{S}(\mathscr{B}(\eta))$ for the different kind of minima $\eta \in \bar{\mathscr{M}}$:

Rectangles of zeroes. We take $\mathscr{B}(\eta) := \bar{B}(\eta)$. The case of a rectangle of zeroes is Ising-like. It is a well known fact (see [Sch1] and [KO]) that if $l^0(\eta) < M^*$, then $\mathscr{S}(\bar{B}(\eta))$ is obtained by setting to minus all but one the sites of one of the shortest sides of the rectangle; hence, $\Gamma(\bar{B}(\eta)) = \varphi_0(l^0(\eta))$. This saddle is downhill connected with (and only with) a smaller rectangle, i.e. a rectangle whose number of zeroes is smaller than that in η.

Moreover, this smaller rectangle has a smaller energy. Hence, this procedure can be iterated and we find a path with cost $\varphi_0(l^0(\eta)) < \gamma_0$ connecting η with $-\underline{1}$.

Irrelevant picture-frames ($l^+ < L^*$). We take $\mathscr{B}(\eta) = \bar{B}(\eta)$. The picture frames differ from other birectangles in having the cost of the shrinking of the minuses as well as the cost of the growth of the pluses much higher than γ_0 (because these moves create a direct interface -1, $+1$). Hence, it is easy to see that $\mathscr{S}(\bar{B}(\eta))$ can be determined by the comparison of the cost γ_0 of the growth of the zeroes and the cost $\varphi_+(l^+)$ of the shrinking of the pluses. For irrelevant picture-frames it is always $\gamma_0 > \varphi_+(l^+)$. Indeed, if $\lambda < 0$ then by (2.20), (2.21), (6.15), (6.4), (6.5) we get that $\gamma_+ > \gamma_0 > \varphi_0(l^0) = \varphi_+(l^+) - 2\lambda(l^+ - \frac{h}{\lambda}) > \varphi_+(l^+)$ (notice that $\{\lambda < 0\} \cap \{l^0 < M^*\} \Rightarrow \{l^+ < L^*\}$ i.e. all picture-frames in $\bar{\mathscr{M}}$ are irrelevant for $\lambda < 0$); for $\lambda > 0$ and $l^+ \leqslant L^*$ by (2.17) we get that $\tilde{L} > L^*$ and hence, by (6.9) $\gamma_0 > \varphi_+(l^+)$. Thus, for irrelevant picture-frames $\mathscr{S}(\bar{B}(\eta))$ is made of configurations obtained from η by setting to zero all but one the spins on one of the shortest sides of the rectangle of pluses. $\mathscr{S}(\bar{B}(\eta))$ is downhill connected with and only with a birectangle $\eta' \in R(L^+(\eta)+2, l^+(\eta)+2, L^+(\eta), l^+(\eta)-1)$ or, in the case $l^+ = 2$, with a rectangle of zeroes $\eta' \in R(L^+(\eta)+2, l^+(\eta)+2, 0, 0)$.

Relevant picture-frames ($l^+ \geqslant L^*$). In this case we take $\mathscr{B}(\eta)$ as the largest transient cycle having as unique ground configuration the configuration η. If $l^+ \geqslant \frac{h}{\lambda}$, $\mathscr{B}(\eta)$ contains the picture-frame η and, possibly, birectangles in the form $R(L^+ + 2, l^+ + 2, L^+ - 1, l^+)$ or $R(L^+ + 2, L^+ + 2, L^+, L^+ - 1)$.

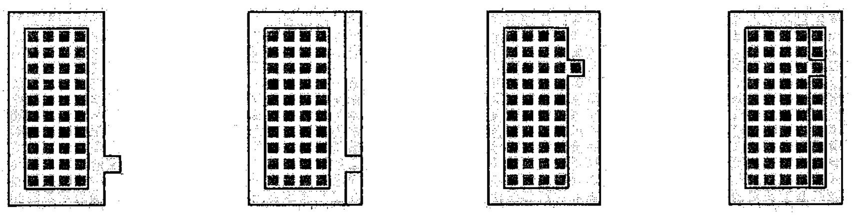
Starting from a relevant picture-frame, we construct a path connecting it with a larger picture-frame and a path connecting it with a smaller picture-frame. We will see that the picture-frame is in $\bar{\mathscr{M}}$ if and only if the latter path has smaller cost with respect to the former one.

- We call *preferred growth path* the following path connecting the picture-frame with a larger picture-frame:

1a) The first step consists in making a zero unit-square protuberance on the longest side of the frame (see Fig. 9). The cost is γ_0.

1b) We continue by filling with zeroes the side to which the protuberance belongs. This part of the path is downhill, the energy gain is $\varphi_0(L^+ + 2)$.

1c) Now having room enough, we form a plus unit-square protuberance at cost γ_+.



1a) 1b) 1c) 1d)

Fig. 9. Preferred growth path.

1d) We end up by filling with pluses the side to which the protuberance belongs. Again, this is downhill.

The maximum of the energy in the preferred growth path is either reached in 1a) or in 1c), the cost being

$$E_g := \gamma_0 + [\gamma_+ - \varphi_0(L^+ + 2)]^+ \tag{6.16}$$

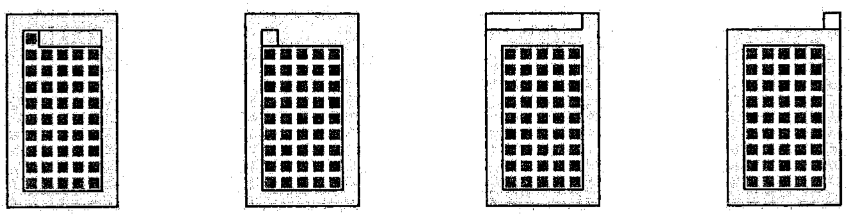
- We construct the *preferred contraction path* in a similar way:

2a) the first step is the erosion of the first $l^+ - 1$ sites of one of the smallest sides of the rectangle of pluses (see Fig. 10), at a cost $\varphi_+(l^+)$.

2b) then, we erase the last site following a downhill path with energy gain γ_+.

2c) we erode the first $l^+ + 1$ sites of one of the smallest sides of the rectangle of zeroes at cost $\varphi_0(l^+ + 2)$ (while continuing the erosion of the rectangle of pluses has higher cost since, for relevant picture-frames with $l^+ \geqslant \frac{h}{\lambda}$, $\varphi_0(l^+ + 2) \geqslant \varphi_+(l^+)$).

2d) finally, we erase the last site in the side following a downhill path.



2a) 2b) 2c) 2d)

Fig. 10. Preferred contraction path.

The maximum of the energy in the preferred contraction path is either reached in case 2a) or in case 2c), the cost is

$$E_c := \varphi_+(l^+) + [\varphi_0(l^+ + 2) - \gamma_+]^+ \tag{6.17}$$

By direct computation we see that

$$E_c < E_g \Leftrightarrow l^+ < f(L^+) \tag{6.18}$$

Indeed, the condition

$$\gamma_0 + \gamma_+ - \varphi_0(L^+ + 2) > \varphi_+(l^+) \tag{6.19}$$

can be expressed in terms of lengths: it is equivalent to

$$l^+ < m^*(L^+) = \tilde{L} + \frac{h-\lambda}{h+\lambda}(\tilde{M} - 2 - L^+) \tag{6.20}$$

Note that $m^*(\tilde{M}-2) = \tilde{L}$, while $m^*(\ell^*) = \ell^*$. Figure 11 a) and b) displays the regions where condition (6.18) is valid if $\tilde{M}-2 > \ell^*$ or $\tilde{M}-2 \leqslant \ell^*$, respectively (note that the region $\tilde{M}-2 \leqslant \ell^*$ is not studied in [CiO]). It is easy to show that the cycle $\mathscr{B}$ containing η and with energy smaller than $H(\eta) + \min\{E_c, E_g\}$ has $F(\mathscr{B}) = \{\eta\}$ and contains at most another local minimum, namely a birectangle strictly smaller than η. Notice that, since $l^+ > L^*$ we always get $F(\mathscr{B}) = \eta$. We refer to [CiO] (where $\bar{B}$ is studied both for picture- frames and for birectangles) for details. By looking at Fig. 11, we note that (for non-squared picture-frames) the preferred growth path is parallel to the ordinate axis (l^+) while the preferred contraction path is parallel to the abscissa axis (L^+). Hence, property (6.18) can be iterated until reaching a wrapping minimum or an irrelevant picture-frame.

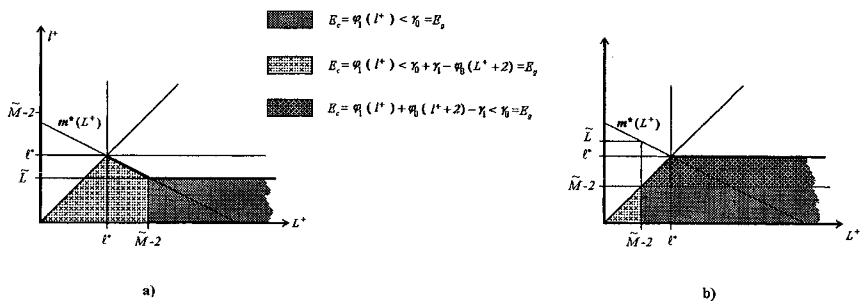


Fig. 11. The contraction path cost is smaller than the growth path cost in the whole region $l^+ < f(L^+) \equiv \min\{\ell^*, \max\{\tilde{L}, m^*(L^+)\}\}$.

In the case $l^+ < \frac{h}{\lambda}$, $\mathscr{B}(\eta)$ can contain many birectangles. This fact can be easily shown by observing that instead of step 2c) in the preferred contraction path it is convenient to continue shrinking the rectangle of pluses. Obviously, all birectangles we can reach in this way are in $\bar{\mathscr{M}}$. Moreover for any of these birectangles, from (6.15), (6.4), (6.5), (6.6), we get $\gamma_0 > \varphi_0(l^0(\zeta_k)) = \varphi_+(l^+(\zeta_k)) - 2\lambda(l^+(\zeta_k) - \frac{h}{\lambda}) > \varphi_+(l^+(\zeta_k)) > \gamma_+$. There are two possibilities: either the path obtained by iterating this procedure can reach a rectangle of zeroes with a cost smaller than $\varphi_0(l^0)$ or it is more convenient to shrink the zeroes by following the preferred growth path (it is immediate to see that the paths erasing the zeroes after having erased more than one row of the rectangle of pluses have higher cost). In any case, we exit from $\mathscr{B}(\eta)$ without creating new non-minuses so that $\Theta(\mathscr{B}(\eta)) < \gamma_0$ and $\mathscr{S}(\mathscr{B}(\eta))$ is downhill connected to a smaller relevant minimum. Hence, the construction can be iterated until reaching a rectangle of zeroes or an irrelevant picture-frame.

Other birectangles. We take $\mathscr{B}(\eta) := \bar{B}(\eta)$. It has been shown in [CiO] that $\mathscr{S}(\bar{B}(\eta))$ can be obtained by one of the following procedures: i) by setting to zero all but one of the spins on one of the shortest sides of the rectangle of pluses (at cost $\varphi_1(l^+)$), ii) by setting to minus all but one of the spins on one of the shortest sides of the rectangle of zeroes (at cost $\varphi_0(l^0)$), iii) by adding a plus unit-square protuberance to the rectangle of pluses (at cost γ_+) and, finally, iv) by adding a zero unit-square protuberance to the rectangle of zeroes (at cost γ_0). Notice that only in the latter case $\mathscr{S}(\bar{B}(\eta))$ is downhill connected with a minimum where the number of minuses decreases (what we call a birectangle larger than η).

• In the region $h > \lambda > 0$, we show that there exists a sequence of birectangles $\{\eta_k\}$, that starts from η and ends in a relevant minimum, such that $\mathscr{S}(\bar{B}(\eta_k))$ is downhill connected with η_{k+1}. The construction of this sequence requires the analysis of a few very similar cases. We report in Table 1, the starting birectangle vs. the ending relevant minimum, the minimum cost of a path reaching this relevant minimum and the total cost of the contraction to minus namely, the minimum cost of the path leading from η to $-\underline{1}$. The last column of Table 1 reports the condition under which the ending relevant minimum is in $\mathscr{G}$. Notice that (by condition 2. and 3. in the definition of $\bar{\mathscr{M}}$) the relevant minimum is in $\bar{\mathscr{M}}$ if and only if $\eta \in \bar{\mathscr{M}}$.

We observe that in the case $l^+ < L^*$ the final picture-frame is always in $\mathscr{G}$; in the case $l^+ \geqslant L^*$, $\max\{l^0, L^+ + 2\} < \tilde{M}$ it is in $\mathscr{G}$ if and only if $l^+ < f(L^+)$; the case $l^+ \geqslant L^*$, $l^0 < \tilde{M} < L^+ + 2$ it is in $\mathscr{G}$ if and only if $l^0 < f(L^+) + 2$; finally, in the case $l^+ \geqslant L^*$, $l^0 \geqslant \tilde{M}$ it is in $\mathscr{G}$ if and only if

Table 1

Starting birectangle	Relevant minimum	Cost of the rel. min.	Total cost of ω: $R(L^0, l^0, L^+, l^+) \to -\underline{1}$	Subcritical condition
$l^+ < L^*$ $\varphi_+(l^+) \leqslant \varphi_0(l^0)$	rectangle $L^0 \times l^0$	$\varphi_+(l^+)$	$\varphi_0(l^0)$	always
$L^+ < L^*$ $\varphi_+(l^+) > \varphi_0(l^0)$	$-\underline{1}$	$\varphi_+(l^+)$	$\varphi_+(l^+)$	always
$l^+ \geqslant L^*$ $\tilde{M} > l^0 > L^+ + 2$	$F(L^+, l^+)$	$\varphi_0(l^0)$	$\varphi_+(l^+)$	$l^+ < f(L^+)$
$l^+ \geqslant L^*$ $\tilde{M} > L^+ + 2 \geqslant l^0$	$F(L^+, l^+)$	$\varphi_0(L^+ + 2)$	$\varphi_+(l^+)$	$l^+ < f(L^+)$
$l^+ \geqslant L^*$ $L^+ + 2 \geqslant \tilde{M} > l^0$	$F(L^+, l^0 - 2)$	γ_+	$\varphi_+(l^+)$	$l^0 < 2 + f(L^+)$
$l^+ \geqslant L^*$ $l^0 \geqslant \tilde{M}$	$F(L^0 - 2, l^0 - 2)$	γ_+	$\varphi_+(l^0 - 2) + \varphi_0(l^0) - \gamma_+$	$l^0 < 2 + f(L^+)$

$l^0 < f(L^0 - 2) + 2 = \ell^* + 2 = f(L^+) + 2$. In the last inequality, we used the fact that $\tilde{M} < m^*(L^+) + 2$ entails $L^+ < \tilde{M} - 2$. As an example, we restrict ourselves to the discussion of the most complicated among the growth and contraction mechanisms leading from a birectangle to a picture-frame: let us consider a birectangle such that $l^+ \geqslant L^*$ and $L^+ + 2 \geqslant \tilde{M} > l^0$ (fifth row in Table 1). It can reach $F(L^+, l^0 - 2)$ at cost γ_+. Indeed, we take $\eta_k := R(L^0 - k, l^0, L^+, l^+)$. $\mathscr{S}(\bar{B}(\eta_k))$ is obtained by setting to minus all but one the sites of one of the shortest sides of the rectangle of zeroes. The depth of $\bar{B}(\eta_k)$ is $\varphi_0(l^0) < \gamma_0$. This sequence ends in $R(L^+ + 2, l^0, L^+, l^+)$. We continue by taking $\eta_{K+k} := R(L^+ + 2, l^0, L^+, l^+ + k)$, where $K := L^0 - L^+ + 2$. In this case, $\mathscr{S}(\bar{B}(\eta_{K+k}))$ is obtained by adding a unit-square protuberance to one of the shortest sides of the rectangle of pluses (there is no room to add it to one of the longer sides without creating a direct interface minus-plus). The depth of $\bar{B}(\eta_{K+k})$ is $\gamma_+ < \gamma_0$.

- The case $h \geqslant -\lambda > 0$, easily leads back to the case $l^+ < L^*$ for $h > \lambda > 0$. Indeed, if $\lambda < 0$ and $l^0 < M^*$ then $l^+ < L^*$. The contraction cost is max $\{(\varphi_0(l^0), \varphi_+(l^+)\}$.

Families of birectangles. The key observation is that if we sequentially shrink all birectangular droplets in η following the above defined pattern, we never form new non-minuses. Hence, following this path, distinct droplets never interact so that we can construct a sequence of minima $\{\eta_k\}$ ending in $-\underline{1}$ and a sequence of cycles $\mathscr{B}(\eta_k)$ in $\mathscr{G}$ with $\mathscr{S}(\mathscr{B}(\eta_k))$

downhill connected to η_{k+1} and $\Gamma(\mathscr{B}(\eta_k)) < \gamma_0$. Thus, a configuration made of birectangles is connected to $-\underline{1}$ at cost smaller than γ_0 if all birectangular droplets in the configuration are in $\mathscr{G}$.

Families of plurirectangles. In all the cases considered, $\Psi(\eta) = \eta$. The case of plurirectangles is slightly less straightforward. If $\varphi_0(l^0(\eta)) > \gamma_1$ the argument is the same as the one we used for birectangles: we find a sequence of minima with the usual properties that goes from the starting configuration η to $\Psi(\eta)$. If $\varphi_0(l^0(\eta)) \leqslant \gamma_1$, it would be natural to shrink first the rectangles of zeroes (until we possibly reach a "hard core" of pluses). We leave to the reader the task of showing that there is no difference between the final configurations obtained by this procedure and by Ψ.

Since $\forall \eta \in \mathscr{G}$ we find a path $\omega: \eta \to -\underline{1}$ with a cost smaller than γ_0, then $\Theta(\mathscr{G}) < \gamma_0$. ∎

In the following Lemma we describe $\mathscr{S}(\mathscr{G})$. The shape and the energy of the saddles coincide with that of the dynamics in [CiO]; this is the most direct way to show that the two dynamics are equivalent.

We call candidate *saddle* one of the following configurations:

$\mathscr{P}_1$: A rectangle $\lceil M^* \rceil \times (\lceil M^* \rceil - 1)$ with a zero unit-square protuberance attached to one of the longest sides.

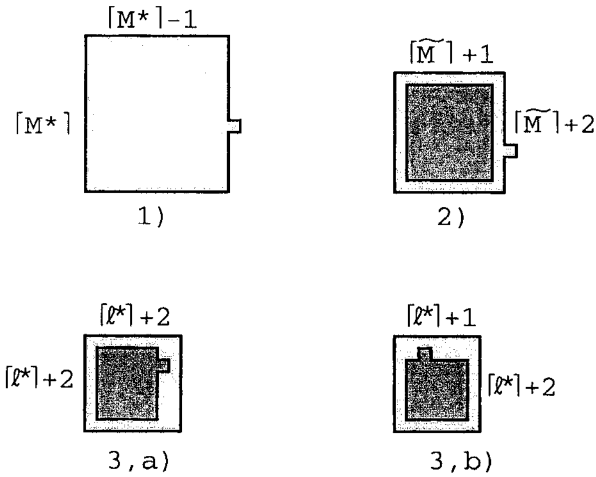


Fig. 12. Candidate saddles.

$\mathscr{P}_2$: A frame $F(\lceil \tilde{M} \rceil, \lceil \tilde{M} \rceil - 1)$ with a zero unit-square protuberance attached to one of the longest sides.

$\mathscr{P}_{3,a,}$: A square $(\lceil \ell^* \rceil + 2) \times (\lceil \ell^* \rceil + 2)$ of zeroes containing in its interior a rectangle $\lceil \ell^* \rceil \times (\lceil \ell^* \rceil - 1)$ of pluses and a plus unit-square protuberance of this rectangle.

$\mathscr{P}_{3,b}$: A rectangle $(\lceil \ell^* \rceil + 2) \times (\lceil \ell^* \rceil + 1)$ of zeroes containing in its interior a square $(\lceil \ell^* \rceil - 1) \times (\lceil \ell^* \rceil - 1)$ of pluses and a plus unit- square protuberance of this square.

Lemma 6.2. $\mathscr{S}(\mathscr{G}) \subset \mathscr{P}_{3,a} \cup \mathscr{P}_{3,b} \cup \mathscr{P}_1$ Moreover, the energy gap between $\mathscr{S}(\mathscr{G})$ and $-\underline{1}$ is $\Gamma_{\oplus}$ defined in (2.18).

Proof. We start showing that $\mathscr{S}(\mathscr{G}) \subset \mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{P}_{3,a} \cup \mathscr{P}_{3,b}$ Direct computation shows that the minimum of the energy of these configurations can only be achieved in $\mathscr{P}_1$, $\mathscr{P}_{3,a}$ or $\mathscr{P}_{3,b}$.

Let $\eta \geqslant \eta'$ be two local minima. By definition of $\bar{\mathscr{M}}$, if $\eta \in \bar{\mathscr{M}}$ then $\eta' \in \bar{\mathscr{M}}$, while if $\eta' \in \mathscr{M} \setminus \bar{\mathscr{M}}$ then $\eta \in \mathscr{M} \setminus \bar{\mathscr{M}}$.

Let $\hat{\eta} \in \mathscr{S}(\mathscr{G})$ and let $\eta \in \mathscr{G}$ be a nearest neighbor configuration of $\hat{\eta}$ differing from it for the spin in x, namely $\hat{\eta} = \eta^{a,x}$.

We denote by $\phi: S \to \mathscr{M}$ the map defined in the following way: given $\xi \in S$, we consider the set I^+ of the pluses in ξ and fill with pluses the bootstrap of this set; then we set to zero all the minuses neighboring the new set of pluses and finally, we set to zero the minuses inside the bootstrap of the resulting set of zero-pluses. It is immediate to show that every $\xi \in S$ is downhill connected with $\phi(\xi)$. Moreover, $\phi(\xi)$ is greater (in the sense of (2.10)) than any other configuration to which ξ is downhill connected. Otherwise, let $\zeta = \phi(\xi)$ and let $\omega_k = \omega_{k-1}^{b,y}$ be the first configuration in a downhill path starting from ξ where there exists a site y with $\omega_k(y) > \zeta(y)$; by the monotonicity of h' appearing in (2.9), we would immediately get $0 \geqslant H(\omega_k) - H(\omega_{k-1}) \geqslant H(\zeta^{b,y}) - H(\zeta)$ contradicting the hypothesis that ζ is a local minimum. Hence, a sufficient condition in order to have $\xi \notin \mathscr{G}$ is $\phi(\xi) \in \mathscr{M} \setminus \bar{\mathscr{M}}$. Also, $\eta = \phi(\eta) \in \bar{\mathscr{M}}$, because $[\phi(\eta)]^{a,x} \geqslant \hat{\eta}$ imply $[\phi(\eta)]^{a,x} \notin \mathscr{G}$ (as $\phi([\phi(\eta)]^{a,x}) \geqslant \phi(\hat{\eta}) \notin \mathscr{G}$), while $\eta < \phi(\eta)$ would imply $H(\phi(\eta)) < H(\eta)$ and, by the monotonicity of h', $H([\phi(\eta)]^{a,x}) < H(\hat{\eta})$ contradicting the hypothesis that $\hat{\eta} \in \mathscr{S}(\mathscr{G})$.

We call *relevant structure* a droplet in the shape of a rectangle of zeroes or of a birectangle with the smallest side-length of the rectangle of pluses larger than L^*.

There are two cases: either $\hat{\eta}(x) = +1$ or $\hat{\eta}(x) = 0$.

1. $\hat{\eta}(x)=0$. Since $\eta \in \bar{\mathcal{M}}$, we observe that there is no downhill path changing the set I^+ of the pluses. Hence, the map ϕ only consists in setting to zero the minuses inside the bootstrap of the non-minuses.

(a) In case η contains many droplets, we use the same argument introduced in [KO]: since the bootstrap procedure is associative with respect to the set union, we can add to x the droplets one by one. We end with two interacting subcritical droplets (see [KO] for details). Since the energy of $\phi(\hat{\eta})$ is larger than the sum of the energy of the droplets, η can only be made of two droplets. Moreover, since the rectangles of pluses are not affected by ϕ, and since rectangles of pluses with $l^+ < L^*$ are irrelevant for determining if η is in $\bar{\mathcal{M}}$ but their presence increases the energy, it is clear that they cannot be present in η. With a similar argument we can conclude that the droplets are in fact relevant structures. Still, these relevant structures are "subcritical" meaning that a configuration obtained from η by cutting down one of the longest sides of the structure has energy smaller than $H(\eta)$. With an argument very similar to the one used in [KO] for the Ising case, we can easily conclude that $\phi(\hat{\eta})$ is "just supercritical," meaning that either $l^0(\phi(\hat{\eta}))=\lceil M^* \rceil$ or $l^0(\phi(\hat{\eta}))=\lceil \tilde{M} \rceil$, i.e. a birectangular droplet that can be turned into a subcritical relevant structure by cutting down one of the longest sides. It was shown in [CiO] that among the configuration $\hat{\eta}$ containing two subcritical relevant structures that give a certain $\phi(\hat{\eta})$, the one with smaller energy is given by a birectangle with zero unit-square protuberance attached.

Direct computation allows to show that all configurations in the shape of a birectangular droplet with $l^+ < f(L^+)$ with a zero unit-square protuberance have higher energy w.r.t. the candidate saddle $\mathcal{P}_2$ while all configurations in the shape of a rectangle of zeroes such that $l^0 < M^*$ with a zero unit-square protuberance have higher energy w.r.t. the candidate saddle $\mathcal{P}_1$. Thus, η cannot contain more than one droplet.

(b) In case η consists of a single droplet, we can use the same reasoning of case 1a) and show that among such configurations the minimum of the energy is achieved either in $\mathcal{P}_2$ or in $\mathcal{P}_1$.

2. $\hat{\eta}(x)= +1$.

(a) In the case there are no direct interfaces in $\hat{\eta}(x)$, we have $\hat{\eta}=\eta^{1,x}$ for some local minimum η and some x in the interior of a rectangle of zeroes in η; hence, is easy to see that there are no downhill paths creating a zero from a minus. The absence of direct interface and $\hat{\eta} \in U(\mathcal{G})$ also imply that η consists of a single rectangle of zeroes containing rectangles of pluses in its interior. Let l^0 and L^0 be the sidelengths of the outer rectangle of zeroes. We observe that the energy of such a configuration can be computed as the sum of the energy formation of the rectangle $l^0 \times L^0$ full of zeroes in the sea of minuses plus the energy formation of the rectangles of pluses in the sea of zeroes. With an argument similar to the one used in case 1a), we can conclude that the shape of pluses inside the rectangle of zeroes must be a rectangle $(\lceil f(L^+(\phi(\hat{\eta})))\rceil - 1) \times L^+$ with a single unit-square protuberance and that $l^0(\phi(\hat{\eta})) = \lceil f(L^+(\phi(\hat{\eta}))\rceil + 2$, $L^0(\phi(\hat{\eta})) = L^+(\phi(\hat{\eta})) + 2$. It is easy to see that among such configurations the one with lower energy is a candidate saddle $\mathscr{P}_{3,a}$, or $\mathscr{P}_{3,b}$ (see proof of Lemma 6.1).

(b) A configuration with a direct interface cannot be in $\mathscr{S}(\mathscr{G})$: indeed, let $e(x)$ be the set of minuses neighboring x in $\hat{\eta}$. Let $\hat{\eta}'$ be the configuration obtained from $\hat{\eta}$ by setting to zero all sites in $e(x)$. Clearly, $H(\hat{\eta}') < H(\hat{\eta})$ while $\hat{\eta}' > \hat{\eta}$. Hence $\hat{\eta}' \notin \mathscr{G}$. It is easy to see that the cost of the path going from η to $\hat{\eta}'$ is lower than the cost to go to $\hat{\eta}$ and, hence, $\hat{\eta} \notin \mathscr{S}(\mathscr{G})$.

Direct computation shows that the minimum of the energy of the candidate saddles has the form (2.18). ▮

We prove in the next Lemma the last requirement of the constructive criterion of Olivieri and Scoppola.

Lemma 6.3. $\mathscr{S}(-\underline{1}, \oplus) = \mathscr{S}(\mathscr{G})$

Proof. Having proved in Lemma 6.1 and in Lemma 6.2 the first two requirements of the criterion introduced in [OS], we have now to show that there always exists a path $\omega: \mathscr{S}(\mathscr{G}) \to \oplus$ with $H(\xi) < H(\mathscr{S}(\mathscr{G}))$ $\forall \xi \in \omega \setminus \mathscr{S}(\mathscr{G})$. It is easy to show, with the methods of the proof of Lemma 6.1, that there always exists an Ising-like path $\omega': \mathscr{P}_1 \to \oplus$ with energy smaller than $H(\mathscr{P}_1)$. This is not the case for $\mathscr{P}_{3,a}$ and $\mathscr{P}_{3,b}$. Indeed, if $\ell^* < l^+ < L^*$, such a path does not exist neither for $\mathscr{P}_{3,a}$, nor for $\mathscr{P}_{3,b}$. However, the region $\ell^* < L^*$ is well inside the region where $\mathscr{S}(\mathscr{G}) = \mathscr{P}_1$. Indeed, direct computation shows that

$$\begin{aligned} \min\{H(\mathscr{P}_{3,a}), H(\mathscr{P}_{3,b})\} &= -2h\lceil \ell^* \rceil^2 + 4\gamma_0 \lceil \ell^* \rceil + 2\gamma_0 \\ &\quad + \min\{(h+\lambda)(\lceil \ell^* \rceil - 1) + 2\gamma_0, h(3\lceil \ell^* \rceil - 2) + \lambda(\lceil \ell^* \rceil + 2)\} \\ &\geqslant -2h\lceil \ell^* \rceil^2 + 4\gamma_0 \lceil \ell^* \rceil + 2\gamma_0 = 2\gamma_0 \ell^* + 2\gamma_0 - 2h\,\delta^2 \end{aligned}$$

where $\delta := \lceil \ell^* \rceil - \ell^*$. On the other hand, if $\ell^* < L^*$,

$$\begin{aligned} H(\mathscr{P}_1) &= -(h-\lambda)\lceil M^* \rceil^2 + (4+h-\lambda)\lceil M^* \rceil - (h-\lambda) \\ &= \gamma_0 M^* - (h-\lambda)(1-\delta+\delta^2) \leqslant \gamma_0 M^* \leqslant \gamma_0(\ell^*+2) \\ &< 2\gamma_0 \ell^* + 2\gamma_0 - 2h\,\delta^2 \end{aligned}$$

where in the second-last inequality we used $\ell^* < L^* \Rightarrow \ell^* > M^* - 2$.

We can then use the construction used in Lemma 6.1: for $\hat{\eta} \in \mathscr{S}(\mathscr{G})$, we set $\eta_1 := \phi(\hat{\eta})$. Than, we take the set $\mathscr{B}_k(\eta_k)$ as the largest metastable cycle having $F(\mathscr{B}_k(\eta_k)) = \eta_k$ and $\eta_{k+1} := \phi(\eta_k)$. Coming back to (6.16) and (6.17), the fact that the shape of the saddle is $\mathscr{P}_{3,a}$ or $\mathscr{P}_{3,b}$ imply that $E_g \leqslant E_c$ for all η_k and thus $\eta_{k+1} > \eta_k$. We can iterate this procedure until we reach a wrapping configuration and, after that, we can easily find a path leading to $\oplus$ at a cost smaller than γ_0. ∎

We end up our analysis with the following key Lemma relating the energy landscape with dynamical aspects:

The following result concerns the finite-volume process $\tilde{\sigma}_{Q,t}$ defined in (6.2) as the restriction to the maximal subcritical cycle introduced in (6.1). We recall that Q is the square of side-length $2D+1$.

Lemma 6.4. Let $\tilde{\tau}^Q_{-\underline{1}} := \min\{t : \tilde{\sigma}_{Q,t} = -\underline{1}\}$ and $\Theta^* := \Theta(\mathscr{G}) \geqslant \Theta(\mathscr{A})$. Let η be a configuration in the cycle $\mathscr{A}$.

For all sufficiently large D, there exists c such that $\forall \varepsilon > 0$

$$\mathbb{P}_\eta(\tilde{\tau}^Q_{-\underline{1}} < e^{\beta(\Theta^*+\varepsilon)}) \leqslant e^{-ce^{\beta\varepsilon}} \tag{6.21}$$

Proof. Let η be a generic configuration in the maximal subcritical cycle $\mathscr{A}$. The probability that the process starting from η reaches a local minimum η' within time 1 undergoing a downhill transition is larger than a constant.

By the definition of Θ^* and Proposition 3.7 in [OS2], the probability of reaching $-\underline{1}$ after time $e^{\beta\Theta^*}$ is uniformly bounded by a constant e^{-c} for any starting configuration $\eta' \in \mathscr{A}$.

We divide the time interval $e^{\beta(\Theta^*+\varepsilon)}$ into smaller intervals of length $e^{\beta\Theta^*}$ and use the Markov property:

$$\mathbb{P}_\eta(\tilde{\tau}^Q_{-\underline{1}} > e^{\beta(\Theta^*+\varepsilon)}) \leqslant (\sup_{\eta' \in \mathscr{A}} \mathbb{P}_{\eta'}(\tilde{\tau}^Q_{-\underline{1}} > e^{\beta\Theta^*}))^{e^{\beta\varepsilon}} \leqslant e^{-ce^{\beta\varepsilon}} \quad ∎ \tag{6.22}$$

6.2. Infinite Volume

We want to first remark that the dynamics used in the present paper and the one of [CiO] have the same speeds of growth (see (2.20), (2.21)); this fact, together with the finite-volume results of the previous subsection, suggest that the two dynamics share the same behavior also at infinite volume.

In this Section $T := e^{\beta k}$ with $k < k_0$. For simplicity, we also assume $k > \gamma_0$ (note that $k_0 > \gamma_0$).

We will follow the scheme of the proof of the Theorem in [DSch2]. First, we will show (Lemmata 6.5, 6.6) that nucleation is a local phenomenon, i.e. if it takes place with the infinite volume dynamics, it must also take place for the process restricted to one of the squares of suitable finite side-length and periodic boundary conditions. This will allow us to use the finite-volume estimates in Lemmata 4.1 and 6.4. Then we use Lemma 6 in [DSch2] to show that the coalescence between supercritical droplets, does not change the asymptotic behavior of $\tau_{\oplus}$. Using the argument of [DSch2] on the existence of a "chronological path," we will estimate from above the speed of growth of a supercritical droplet by $e^{-\beta\gamma_0/2}$, showing that, with high probability, the origin cannot be $\oplus$-infected within T.

Lemma 6.5. Let $\Lambda_1 \subset \Lambda_2 \subset \mathbb{Z}^2$. If $\exists x \in \Lambda_1$ such that $\sigma_{\Lambda_1;t}(x) \neq \sigma_{\Lambda_2;t}(x)$, then $\mathscr{C}^{\oplus}_{x,t}(\sigma_{\Lambda_2}, t) \cap \Lambda_1^c \neq \varnothing$, That is, there exists a space-time cluster that connects (x, t) with $(\Lambda_1^c, 0)$.

Proof. Ab absurdo, let $s = t_x^{a,b}$ be the first time in which the hypothesis is satisfied but the thesis is not. Let $s^- < s$ be sufficiently large (such that there are no spin flips between s^- and s). At time s^- none of the nearest neighbors of x can be $\oplus$-connected with Λ_1^c, otherwise (x, s) would also be $\oplus$-connected with Λ_1^c. Since x is the first site not $\oplus$-connected with Λ_1^c whose spin is different in the two dynamics, it follows that x as well as all its nearest neighbors have the same spin in both processes. Hence, the rates for the site x are the same in both processes, but this is absurd since it contradicts the assumption that at time s the two processes differ. ∎

The following key Lemma allows us to reduce nucleation to a finite size phenomenon, saying that, at times shorter than T, if nucleation does not take place, then there are no space-time clusters wider than a constant D'.

Lemma 6.6. Let $\bar{\tau}^{Q}_{>D} := \min\{t : \exists \mathscr{C}^{\oplus}(\bar{\sigma}_Q, t) \text{ with } |\mathscr{C}^{\oplus}(\bar{\sigma}_Q, t)| > D'\}$ and

$$\bar{\tau}^{Q}_{\text{nucl}} := \min\{t : \bar{\sigma}_{Q;t} \in \partial^{+}\mathscr{A}\} \tag{6.23}$$

Then

$$\mathbb{P}(\bar{\tau}^{Q}_{>D} < \min\{\bar{\tau}^{Q}_{\text{nucl}}, T\}) \leqslant e^{-2\beta k_0} \tag{6.24}$$

for D and D' large enough.

Proof.

$$\mathbb{P}(\bar{\tau}^{Q}_{>D} < \min\{\bar{\tau}^{Q}_{\text{nucl}}, T\}) \leqslant \mathbb{P}(\tilde{\tau}^{Q}_{>D} < T) \tag{6.25}$$

where $\tilde{\tau}^{Q}_{>D} := \min\{t : \exists \mathscr{C}^{\oplus}(\tilde{\sigma}_Q, t) \text{ with } |\mathscr{C}^{\oplus}(\tilde{\sigma}_Q, t)| > D'\}$

Lemma 6.4 claims that starting from every configuration $\sigma \in \mathscr{A}$, the probability that $\tilde{\sigma}$ stays out of $-\underline{1}$ for a time $e^{\beta(\Theta^* + \delta)}$ tends to zero superexponentially fast for any $\delta > 0$.

On the other hand, in order to form a space-time cluster of width larger than D', it must happen either at least $D'/2$ times that a minus site changes to zero when it has at most one nearest neighbor zero, or at least $D'/4$ times that a direct interface -1, 1 is formed. Therefore, it must happen at least $D'/4$ times that a Poisson clock with rate 1 rings before $e^{\beta(\Theta^* + \delta)}$ and that a random variable uniformly distributed in $[0, 1[$ is smaller than $e^{-\beta\gamma_0}$. It follows that

$$\mathbb{P}(\tilde{\tau}^{Q}_{>D} < T) \leqslant T D^2 (e^{-\beta\gamma_0} e^{\beta(\Theta^* + \delta)})^{D'/4} \leqslant e^{-2\beta k_0} \tag{6.26}$$

for sufficiently large D', and small δ. ∎

The following Lemma estimates the nucleation probability in Λ^0 by using the finite volume results of Lemma 6.4 and the locality results of Lemma 6.6. We recall that $\tau^{\Lambda^0}_{\text{LECE}}$ was defined in (2.36).

Lemma 6.7. Let $\bar{\tau}^{\Lambda^0}_{>D} := \min\{t : \exists \mathscr{C}^{\oplus}(\bar{\sigma}_Q, t) \text{ with } |\mathscr{C}^{\oplus}(\bar{\sigma}_Q, t)| > D'\}$. $\forall\, \delta > 0$

$$\mathbb{P}(\min\{\bar{\tau}^{\Lambda^0}_{>D}, \tau^{\Lambda^0}_{LECE}\} < T) \leqslant e^{-\beta(k_0 - k - \delta)} \tag{6.27}$$

Proof. By Lemma 6.5, the configuration inside a square $Q^* := Q(D')$ can only be influenced by a suitable space-time cluster. Lemma 6.6 states that (with overwhelming probability) for subcritical configurations, this space-time cluster is smaller than D'. Hence, having a nucleation or a large space-time cluster in Λ^0 implies a nucleation or a large space-time cluster in the dynamics restricted to one of the translates of $Q := Q(5D')$ for the process $\bar{\sigma}_{Q+x,t}$.

$$\mathbb{P}(\tau^{\Lambda^0}_{\text{LECE}} < T) \leqslant |\Lambda^0| \; (\bar{\tau}^Q_{\text{nucl}} \leqslant T) + \mathbb{P}(\bar{\tau}^Q_{>D} < \max\{\bar{\tau}^Q_{\text{nucl}}, T\})) \tag{6.28}$$

$$\leqslant e^{2\beta(k_0-\gamma_0/2)}(e^{-\beta(\Gamma_\oplus - k - \delta/2)} + e^{-2\beta k_0}) \tag{6.29}$$

and then the thesis. ∎

The following Lemma states that with large probability the droplet that infects the origin does not come from too far.

Lemma 6.8. $\forall t \leqslant T$,

$$\mathbb{P}(\sigma_t(0) \neq \sigma_{\bar{\Lambda};t}(0)) \to 0 \tag{6.30}$$

Proof. The proof is standard: it is a large deviation estimate based on the fact that the growth speed is at most 1 (we recall that $\bar{\Lambda} := Q(e^{\beta k_0})$). ∎

Proof of Proposition 1 part a). In order to prove Proposition 1, we will first make use of the result of Lemma 6 in [DSch2] to show that the origin is not infected by a large- scale coalescence phenomenon; then, we will show that it is also very unlikely that the origin is infected by the growth of an isolated droplet.

The bootstrap procedure defined in Subsection 2.8 will be used to estimate the possible effects of the coalescence of supercritical droplets.

Let $\Lambda^0_j := Q(3e^{\beta(k_0-\gamma_0/2)}) + j(1 + 2\lfloor e^{\beta(k_0-\gamma_0/2)} \rfloor)$, with $j \in \mathbb{Z}^2$. The Λ^0_j are partially overlapping squares covering $\bar{\Lambda}$. Following [DSch2], we define a process on the rescaled lattice of side-length equal to the side-length of Λ^0:

$$v_j := \mathbf{I}\{\min\{\tau^{\Lambda^0_j}_{\text{LECE}}\} \leqslant T\} \tag{6.31}$$

In the finite volume $\bar{\Lambda}$, we apply the bootstrap procedure on the pluses in v and obtain the configuration

$$v = \mathbf{B}^+(v) \tag{6.32}$$

By taking the results of Lemmata 6.7 and 6.8, we can use Lemma 6 in [DSch2] (that is a variation of the Theorem in [AL] to include finite range of dependence) and conclude that

$$\mathbb{P}(v_0 = 1) \to 0 \tag{6.33}$$

The physical meaning of (6.27) and (6.33) is that the zero-pluses can neither be created inside Λ^0 nor come from the outside because of the coalescence between supercritical droplets. Indeed, we will show that when starting from a configuration with pluses in the whole region where $v_j = 1$, it is very unlikely that supercritical droplets can coalesce. To conclude the proof of Proposition 1 part a), we need to estimate the speed of growth of an isolated droplet, that could possibly be increased by the coalescence with subcritical droplets. This typical speed will turn out to be so small that neither the growth of an isolated supercritical droplet can reach the origin nor the coalescence of supercritical droplets into $\bar{\Lambda}$ can take place.

Let $\mathscr{D} := \{x \in \mathbb{Z}^2$ such that $v_j = 1$, for any j such that $x \in \Lambda_j^0\}$ and $\bar{\mathscr{D}} := \{y \in \mathbb{Z}^2$ such that $\exists x \in \mathscr{D}$ with $\|x - y\| \leqslant \frac{1}{3}(1 + 2\lfloor e^{\beta(k_0 - \gamma_0/2)} \rfloor)\}$.

We call ρ the configuration on the original lattice induced by v in the following way: $\rho(x) := +1$ if $x \in \mathscr{D}$, $\rho(x) := 0$ if $x \in \partial^+\mathscr{D}$, $\rho(x) = -1$ otherwise.

Following again [DSch2], we define, on the probability space of the Poisson times, the process

$$\xi_t := \sigma^{\rho}_{\bar{\Lambda};\, t} \tag{6.34}$$

Since it uses the information at time T, this process is clearly non-Markov.

From the first of (2.14), it follows that

$$\xi_t \geqslant \sigma_{\bar{\Lambda};\, t} \tag{6.35}$$

The idea behind this process ξ_t is that all supercritical droplets were previously considered in ρ; since the distance between such droplets is at least $e^{\beta(k_0 - \gamma_0/2)}$, with high probability they can only coalesce with subcritical droplets before leaving $\bar{\mathscr{D}}$. By Lemma 6.6, these subcritical droplets can at most measure D'.

Let us consider the picture-frame envelope (see Fig. 5) of the section at time s of a space-time cluster.

Consider the only supercritical droplet $\mathscr{D}_0(x) \subset \mathscr{D}$ at a distance $\frac{1}{3} e^{\beta(k_0 - \gamma_0/2)}$ from x and the section $\mathscr{D}_t(x)$ at time t of its space time cluster $\mathscr{C}^{\oplus}_{x_0,\, 0}(\xi, t)$, where $x_0 \in \mathscr{D}_0(x)$.

The key argument of the proof is that all three possible growth mechanisms that cause the cluster to exit the picture-frame envelope (i.e. by creation of unit-square protuberances of zeroes inside the minuses, by creation of a direct interface minus-plus and by coalescence with subcritical droplets) have finite range in the sense that they give rise to a maximal growth of 1, 2 or D', respectively . Indeed, the argument used in Lemma 8 in [DSch2] allows to claim that if $y \notin \mathscr{D}_t(x)$ and the distance between y and $\mathscr{D}_0(x)$ is smaller than $\frac{1}{3} e^{\beta(k_0 - \gamma_0/2)}$, then $|\mathscr{C}^{\oplus}_{y,\, t}(\xi, t)| \leqslant D'$.

We call *special point* a space-time point (x, t) having one of the following characteristics:

1. $\sigma_t(x)$ changes from minus to zero having only one neighbor zero,

2. $\sigma_t(x)$ changes increasing the length of the direct interface minus-plus,

3. $\mathscr{C}^{\oplus}_{x,t-\varepsilon}(\xi, t)$ (for sufficiently small ε) becomes connected to a supercritical droplet at time t because of the spin change on a different site (closer than $D'+1$ to x).

Let x be the first site outside $\bar{\mathscr{D}}$ that becomes non-minus and let $\vec{\alpha}$ be the unitary vector, parallel to one of the axes, such that $\|x-y\| = \frac{1}{3}(1+2\lfloor e^{\beta(k_0-\gamma_0/2)}\rfloor)$ for some $y \in \mathscr{D}_0$ (it is clear that it is sufficient to consider sites for which $\vec{\alpha}$ is unique because a corner in $\bar{\mathscr{D}}$ cannot be the only first site in $\partial\mathscr{D}$ becoming non-minus).

We call *chronological path* a sequence of space-time points (x_i, t_i) such that $\xi_{t_i}(x_i) \neq -1$, $t_{i+1} > t_i$ and $0 < (x_i - x_{i+1}) \cdot \vec{\alpha} \leqslant 2D'+1$.

It is easy to see the existence of a chronological path made of special points connecting x with $\mathscr{D}_0$. Indeed, it is possible to number all special points involving advancing along $\vec{\alpha}$: t_{k+1} is chosen as the first time after t_k such that (x_{k+1}, t_{k+1}) is a special point with $(x_{k+1}-x_k)\cdot\vec{\alpha} > 0$. In this way, the modulus of the projection along $\vec{\alpha}$ of the vector joining two consecutive points is smaller than $2D'+1$. Indeed, if $\mathscr{D}_{t_{k+1}}$ is not contained into the picture-frame envelope of $\mathscr{D}_{t_k}$, then there must be a special point in the appropriate direction. The resulting chronological path $\{(x_0, t_0), \ldots, (x_n, t_n)\}$ has $x_n = x$, $x_0 \in \mathscr{D}_0(x)$, $n \geqslant N := Ce^{\beta(k_0-\gamma_0/2)}$.

By using a finite speed argument, it is easy to show that with large probability one of these paths must have $t_n \leqslant C'T$.

In order to give a bound on the probability of following within time T a chronological path with given sites, we split the estimate into two cases: either there are more than $N/2$ special points of kind 1) and 2) in the chronological path, or there are more than $N/2$ special points of kind 3).

The first case is a standard large deviation estimate: since the minimum rate for special points of kind 1) and 2) is $e^{-\beta\gamma_0}$, the probability of following the chronological path is lower than

$$\left(\frac{C''e^{-\beta\gamma_0}T}{N}\right)^{N/2} \tag{6.36}$$

The second case of many subcritical droplets is more complicated. Let $\{(x_{k_i}, t_{k_i})\}_i$ be the sub-path of points of kind 3).

We split the proof in two cases:

a) more than $N/4$ points are such that $t_{k_i} > t_{k_{i-1}}$;
b) more than $N/4$ points are such that $t_{k_i} = t_{k_{i-1}}$.

Case a) is again a large-deviation estimate: Lemma 6.5 gives the independence among the points in the sub-path, while the formation rate of an isolated zero in the minuses is $e^{-\beta(4-h+\lambda)}$. Hence, the probability of case a) can be bounded as

$$\left(\frac{e^{-\beta(4-h+\lambda)}T}{N}\right)^{N/4} \tag{6.37}$$

The probability of case b) can be estimated as

$$(e^{-\beta(4-h+\lambda-\varepsilon)})^{N/4} \tag{6.38}$$

where ε is an arbitrarily small constant. Indeed, (as a consequence of Lemma 6.5) the probability that at time t_{k_i} there exists in $Q+x_{k_{i+1}}$ a space-time cluster smaller than D for the process ξ_t is bounded by $\mathbb{P}_{-\underline{1}}(\tilde{\sigma}_{Q;\,t} \neq -\underline{1})$.

As usual, the probability of the event $\{\tilde{\sigma}_{Q;\,t}$ does not visit $-\underline{1}$ for a time $e^{\beta\gamma_0}\}$ tends to zero superexponentially fast.

We split the event $\{\exists s < e^{\beta\gamma_0} : \{\tilde{\sigma}_{Q;\,t} = -\underline{1}\} \cap \{\tilde{\sigma}_{Q;\,t-s} = -\underline{1}\}\}$, into two cases: either the process exits the strict basin of attraction of $-\underline{1}$ or not.

The probability of the first event can be immediately bounded by $e^{-3\beta\gamma_0}$. To estimate the second case, we use the fact that the probability that the process does not visit $-\underline{1}$ for a time $e^{\beta\varepsilon'}$ tends to zero superexponentially fast. Since the probability that the process exits form $-\underline{1}$ in a time $e^{\beta\varepsilon'}$ can be bounded by $e^{-\beta(4-h+\lambda-\varepsilon)}$, we can put together the four cases and get the bound $\mathbb{P}_{-\underline{1}}(\tilde{\sigma}_{Q;\,t} \neq -\underline{1}) \leqslant e^{-\beta(4-h+\lambda-\varepsilon)}$.

The sites of chronological paths with $x_n = x$, $x_0 \in \mathscr{D}_0(x)$, with $n \geqslant N$, $t_n \leqslant C'T$ and containing at least $N/2$ special points of a given kind can be arranged in no more than

$$T\left(\frac{C'T}{N}\right)^{N/2} = T(Ce^{-\beta(k_0-k-\gamma_0/2)})^{N/2} \tag{6.39}$$

ways (see [KeSch] and [DSch2] for details).

We can conclude the proof of Proposition 3.1 case a) by multiplying (6.39) by the sum of (6.36), (6.37) and (6.38) and by $|\partial^+\bar{\mathscr{D}}| \leqslant |\bar{\Lambda}|$:

$$\begin{aligned}\mathbb{P}(\tau_\oplus < T) \leqslant T(Ce^{-\beta(k_0-k)}e^{\beta(\gamma_0/2)})^{N/2}\,(Ce^{-\beta(k_0-k-\gamma_0/2)})^{N/2}\,(e^{2\beta k_0})\\ + \mathbb{P}(v_0 = 1) \to 0\end{aligned} \tag{6.40}$$

7. PROOF OF PROPOSITION 3.1) CASE b) (RELAXATION TO $\oplus$)

In this Section $T = e^{\beta k}$ with $k > k_0$. For sake of simplicity, we also assume $k < \Gamma_{\oplus}$. Indeed, the proof of Proposition 1 case b) is immediate for $k \geqslant \Gamma_{\oplus}$ since a finite-volume relaxation mechanism is active on this scale of time.

The strategy of the proof will be the following: first, we extend the finite volume estimates of [CiO] to times much shorter than the characteristic nucleation time $e^{\beta \Gamma_{\oplus}}$ by using Lemma 4.3. In Lemma 7.1 we prove that there must be in Λ^0 a site x such that there is a nucleation before T for the process $\sigma_{Q+x;\,t}$; then, in Lemma 7.2 we will prove that with overwhelming probability the $\oplus$-infection process is irreversible. Eventually, we will use the estimates on the spreading speed of the $\oplus$-infection obtained in Lemmata 7.3 and 7.4, in a construction similar to the one used in Lemma 1 in [MO] to show that a "non-minus" droplet is likely to infect the origin within T.

Let us now consider $\sigma_{Q+x;\,y}$. Since D is large enough, we see that the saddle between $-\underline{1}$ and $\oplus$ with minus boundary conditions has the same shape and energy $\Gamma_{\oplus}$ as the critical droplet with periodic boundary conditions.

Similarly, it is possible to prove that the largest inner resistance of the largest cycle containing $-\underline{1}$ and not intersecting $\oplus$ is lower than or equal to the resistance $\Theta^* < \gamma_0$ that we have with periodic boundary conditions.

By using Proposition 3.7 in [OS] and Lemma 4.3, we get

$$\mathbb{P}(\check{\tau}_1^{\oplus}(x) < T) \geqslant e^{-\beta(\Gamma_{\oplus} - k + \delta)} \tag{7.1}$$

for sufficiently large β.

Lemma 7.1. Let $\hat{\Lambda} := \Lambda(\frac{1}{2}\, e^{\beta(k_0 - \gamma_0/2)})$

$$\mathbb{P}(\min_{x \in \hat{\Lambda}} \check{\tau}_1^{\oplus}(x) < T) \to 1 \tag{7.2}$$

Proof. We tile $\hat{\Lambda}$ with squares of the form $Q + (2D+3)\, j$ and use the first inequality in (2.14) to set minus boundary conditions. We get

$$\mathbb{P}(\min_{x \in \Lambda^0} \check{\tau}_1^{\oplus}(x) < T) \geqslant 1 - (\mathbb{P}(\check{\tau}_1^{\oplus}(x) > T))^{ce^{2\beta(k_0 - \gamma_0/2)}} \geqslant 1 - e^{-Ce^{\beta(k - k_0 + \delta)}} \quad \blacksquare \tag{7.3}$$

The following key Lemma represents the irreversibility of the $\oplus$-infection process.

Lemma 7.2. Let $\mathscr{E}$ be the event $\{\min_{x \in \Lambda^0}(\hat{\tau}_1^{\oplus}(x)) \geqslant e^{\beta(k_0+k_+)}\}$. We have:

$$\mathbb{P}(\mathscr{E}) \to 1 \tag{7.4}$$

Proof. We can give the estimate of $\hat{\tau}_1^{\oplus}(x)$ by considering an initial condition without minuses (we recall that, by definition, $\hat{\tau}_1^{\oplus}(x) > \check{\tau}_1^{\oplus}(x)$). We take advantage of the fact that the $\oplus$-infected configurations have a finite probability of reaching a configuration without minuses within time 1. Let us consider the process $\sigma_{Q+x;\,t}$ restricted to $Q+x$ with minus boundary conditions. When starting from a configuration having at most $D/3$ minuses, there is always a downhill trajectory that erodes the minus droplets from the corners by substituting the minus spins with zeroes. The probability that the time elapsing between the last visit to "non-minus" and $\hat{\tau}_1^{\oplus}$ is larger than $e^{\beta\delta}$ is super-exponentially small in β (see e.g. proof of Lemma 4.3).

On the other hand, there is no downhill path creating minuses outside the region spanned by the bootstrap of the minuses (see Fig. 8). Then, in order to create $D/3$ minuses, the process must go against the drift at least $\frac{1}{3}\sqrt{D/3}$ times. Indeed, let $\omega\colon \oplus \to \eta$, where η contains more than $D/3$ minuses. We extract a path ω' from ω, by picking out the moves increasing the total number of minuses.

Let $\mathscr{N}_\eta$ be the sum of the largest sides of the rectangles in the set obtained by the bootstrap of the minuses.

Since every minus added with energy gain does not increase $\mathscr{N}_{\omega'_k}$ while a minus added with energy loss can at most increase $\mathscr{N}_{\omega'_k}$ by 3, it is easy to prove that at least $\frac{1}{3}\lceil \sqrt{(D/3)}\,\rceil$ moves against the drift are needed to create $D/3$ minuses.

Hence, it must happen at least $\frac{1}{3}\sqrt{D/3}$ times that a Poisson clock with rate 1 rings within $e^{\beta\delta}$ and that a random variable uniformly distributed in $[0, 1]$ is smaller than $e^{-\beta\gamma_0}$. It follows that

$$\mathbb{P}(\check{\tau}_1^{\oplus}(x) < e^{\beta(k_0+k_+)}) \leqslant C e^{\beta(k_0+k_+)} (e^{-\beta(\gamma_0-\delta)})^{1/3\sqrt{D/3}} \leqslant |\Lambda^0|^{-1} e^{-\beta\delta'} \tag{7.5}$$

for sufficiently large D.

Multiplying by $|\Lambda^0|$, we get the thesis. ∎

The following Lemmata 7.3 and 7.4 estimate the spreading speed of the $\oplus$-infection. These results will be used in the proof of Lemma 7.5.

Lemma 7.3. Let

$$\zeta_2(x) := \check{\tau}_1^{\oplus}(x) - \min\{t : \exists y, y' \text{ with } y \neq y';$$
$$\|x-y\| = \|x-y'\| = 1 \text{ and } \check{\tau}_1^{\oplus}(y) \leqslant t, \check{\tau}_1^{\oplus}(y') \leqslant t\} \tag{7.6}$$

be the time needed to $\oplus$-infect the site x, once it has two $\oplus$-infected neighbors.

Let

$$\mathscr{E}_2 := \{\forall x \in \Lambda^0, \zeta_2(x) < e^{2\beta\delta}\} \tag{7.7}$$

Then, $\forall\, \delta > 0$, for large β,

$$\mathbb{P}(\mathscr{E} \cap \mathscr{E}_2) \geqslant 1 - e^{-e^{\beta\delta}} \tag{7.8}$$

Proof. We use the same argument we used in the proof of Lemma 6.6.

It is clear that once x has two $\oplus$-infected neighbors, $\mathscr{E}$ implies that before $e^{\beta(k_0+k_+)}$ the number of minuses into $Q+x$ is at most $\frac{2}{3}D+1$.

By using the definition of $\check{\tau}_1^{\oplus}$ and the first inequality in (2.14), we can restrict ourselves to considering the process $\sigma^{\eta}_{Q+x;\,t}$ with the initial condition η containing at most $\frac{2}{3}D+1$ minuses.

Again, the probability of staying out of local minima for a time $e^{\beta(\delta/2)}$ is superexponentially small in β:

By Proposition 3.7 in [OS],

$$\mathbb{P}(\{\check{\tau}_1^{\oplus}(x) > s + e^{\beta(\delta/2)}\} \cap \mathscr{E}) \leqslant e^{-c} \tag{7.9}$$

By using the Markov property, we get

$$\mathbb{P}(\{\check{\tau}_1^{\oplus}(x) > s + e^{2\beta\delta}\} \cap \mathscr{E}) \leqslant e^{-ce^{3\beta/2\delta}} \tag{7.10}$$

in a standard way.

By multiplying (7.10) by the number of sites and by $e^{2\beta\delta}$, we get the thesis. ∎

Lemma 7.4. Let us consider the sub-lattice $\mathbb{Z}'^2 := (3D\mathbb{Z})^2$ of side-length $3D$.

Let $\mathbb{A} := \{A_i\}_i$ be a partition of $\Lambda^0 \cap \mathbb{Z}'^2$ into sets A_i.

Let

$$\zeta_1(i) := \min_{x \in A_i}(\check{\tau}_1^{\oplus}(x) - \min_{y_x : \|x-y_x\|=1} \check{\tau}_1^{\oplus}(y_x)) \tag{7.11}$$

$$\mathscr{E}_1(i) := \left\{\zeta_1(i) < \frac{e^{\beta(\gamma_0+2\delta)}}{|A_i|}\right\} \tag{7.12}$$

and

$$\mathscr{E}_1 := \bigcap_i \mathscr{E}_1(i) \tag{7.13}$$

Then, $\forall\, \delta > 0$, for large β,

$$\mathbb{P}(\mathscr{E} \cap \mathscr{E}_1) \geqslant 1 - e^{-e^{\beta\delta}} \tag{7.14}$$

Proof. Again, we use minus boundary conditions to get the bound we need: this gives independence among the $\oplus$-infection times in the sites in $\mathbb{Z}'^2$.

After the infection of one of the neighbors of x, $\mathscr{E}$ implies that the number of minuses in $Q+x$ is at most $\frac{4}{3}D$. We can therefore consider an initial condition such that the number of minuses in each $Q+x$ is at most $\frac{4}{3}D$.

Since the energy gap with a configuration without minuses is at most γ_0 (i.e. the energy of a zero unit-square protuberance in the only row or column possibly full of minuses), we can use Lemma 4.3 with $\Theta = 0$: $\forall x \in A_i$, $\forall t \leqslant e^{\beta(k_0+k_+)}$

$$\mathbb{P}((\check{\tau}_1^{\oplus}(x) < t) \mid \mathscr{E}) \geqslant te^{-\beta(\gamma_0+\delta)} \tag{7.15}$$

thus

$$\mathbb{P}(\{\min_{x \in A_i} \check{\tau}_1^{\oplus}(x) > t\} \mid \mathscr{E}) \leqslant (1 - te^{-\beta(\gamma_0 + (3\delta/2))})^{|A_i|} \leqslant e^{-te^{-\beta(\gamma_0+(3\delta/2))}|A_i|} \tag{7.16}$$

If $t = e^{\beta(\gamma_0+\delta)}/|A_i|$, multiplying by $|\Lambda^0|$ we get the thesis. ∎

We can now use the shell construction used in [MO] to prove Lemma 7.5.

This Lemma exhibits a growth mechanism by squares of "non-minuses." This takes place at speed $e^{-\beta\gamma_0}$ until the droplet reaches the critical size of $e^{\beta(\gamma_0/2)}$ and afterwards at speed $e^{-\beta(\gamma_0/2)}$.

The fact that this mechanism infects the origin within time T, concludes the proof of Proposition 1, b). The result will also be used in the proof of Proposition 1, d).

Lemma 7.5. Let us consider an initial condition without minuses inside $Q+x$ for a site x in $\Lambda(\frac{1}{2}\, e^{\beta(k_0-(\gamma_0/2))})$.

Let $M > \frac{\gamma_0}{2}$, $L := \lfloor e^{\beta M} \rfloor$ and $k(M) := M + \frac{\gamma_0}{2}$; then $\forall k' > k(M)$

Then, in the event $\mathscr{E} \cap \mathscr{E}_1^c \cap \mathscr{E}_2^c$ we have

$$\max_{y \in Q(L)} \{\check{\tau}_1^{\oplus}(y)\} < e^{\beta(k(M)+\delta)} \qquad \forall\, \delta > 0$$

Proof. Let $\ell_0 := e^{\beta(\gamma_0/2)}$.

Let $Q_i := Q(i)$ be the square of side-length $2i+1$ centered in x and let $\partial Q_i := Q_i \setminus Q_{i-1}$. We denote by $\mathscr{C}_i$ the set of the four corners of Q_i. In order to estimate the growth time of Q_i we split ∂Q_i into suitable intervals A_i^h (horizontal or vertical).

An efficient way to fill a shell ∂Q_i larger than ℓ_0 is to divide it into intervals A_i^h of length of order ℓ_0 and wait until a first site is occupied by a non-minus in each of them. All other sites can be subsequently occupied at rate 1.

Let $n_i := 4\lfloor \frac{2i}{\ell_0} \rfloor$. We take the N side of ∂Q_i with the NE corner but without the NW one and split this set into $n_i/4$ intervals A_i^h: if $2i > \ell_0$ we take the first $n_i/4 - 1$ intervals of length to and the last interval with length between to and $2\ell_0 - 1$; we continue clockwise by sequentially partitioning the other sides without the previously considered corners (see Fig. 11) with the same criterion; similarly, if $2i \leqslant \ell_0$, then $n_i = 4$. In this case A_i^h can be seen as one of the four sides of Q_i (without some of the corners in $\mathscr{C}_i$).

Let us set $\hat{A}_i^h := A_i^h \setminus \mathscr{C}_i$.

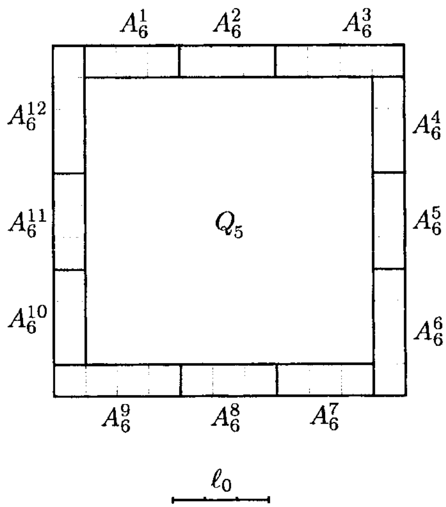


Fig. 13. Shell construction.

The proof is now reduced to the deterministic computation of the time needed to $\oplus$-infect Λ^0. Since the time needed to $\oplus$-infect the i-TH shell is bounded by

$$\max_{h \leqslant 4n_i} \left(|A_i^h| \, e^{\beta(\delta/2)} + \frac{1}{\min_{h \leqslant 4n_i} |A_i^h|} \, e^{\beta(\gamma_0 + (\delta/2))} \right) \tag{7.17}$$

the time needed to $\oplus$-infect $\Lambda(L)$ can be bounded by

$$\sum_{i=1}^{L} \left(\max_{h \leqslant 4n_i} |A_i^h| \, e^{\beta(\delta/2)} + \frac{1}{\min_{h \leqslant 4n_i} |A_i^h|} \, e^{\beta(\gamma_0 + (\delta/2))} \right) \tag{7.18}$$

$$\leqslant C e^{\beta(\delta/2)} \left(L\ell_0 + e^{\beta\gamma_0} \ln \ell_0 + \frac{L}{\ell_0} \, e^{\beta\gamma_0} \right) \tag{7.19}$$

$$\leqslant e^{\beta(M + (\gamma_0/2) + \delta)} \quad \blacksquare \tag{7.20}$$

8. PROOF OF PROPOSITION 3.1) CASE c)

Case c) of Proposition 1 is a corollary of the Theorem in [DSch2].

Indeed, by restricting the dynamics to the configurations in $\{0, 1\}^{\mathbb{Z}^2}$, the system is reduced to the dynamic Ising model $\mathscr{I}^+$ with magnetic field $h + \lambda$.

On the other hand, by using the first inequality in (2.14), it is easy to see that the hitting time to a configuration having the origin $+$infected is not larger than the one we get starting from $\underline{0}$ and preventing transitions from 0 to -1.

9. PROOF OF PROPOSITION 3.1) CASE d)

In this Section $T := e^{\beta k}$ with $k > k^+$. For simplicity we take $k < \Gamma_+$, being the case $k \geqslant \Gamma_+$ a trivial consequence of the finite-volume results in [CiO].

Let $\gamma := \max \{\gamma_0, \gamma_+\} = 2 - h + |\lambda|$.

Note that, while in the region $\lambda < 0$, $k_* = k_+$, in the region $\lambda < 0$ a mixed effect shows up and the relaxation time $\tau_{\oplus +}$ depends on the nucleation rate of the pluses as well as on the growth rate of the zeroes.

The proof of Proposition 1) case d) is analogous to the one of case b). Yet in this case, since the growth of pluses in the sea of minuses involves the zeroes (the direct interface is too expensive), the growth speed of a plus droplet is $e^{\beta(\gamma/2)}$.

The strategy of the proof is the following:

By Lemma 7.5, we know that, with overwhelming probability, within a time $e^{\beta(k_+ +\delta)}$ after $\tau_{\oplus}$, there is a $\oplus$-infected square of side-length $e^{\beta(k_+ -(\gamma_0/2))}$ centered in the origin. In Lemmata 9.3, 9.4 and 9.5, we show that the spreading of the $+$infection into the $\oplus$-infected sites takes place at the same rates we would have if -1 is not allowed (hence with spreading speed $e^{-\beta(\gamma_+/2)}$).

We will then use Lemma 9.6 to conclude the proof of Proposition 1.

Remark . From Lemmata 7.2, 7.3, 7.4 and 7.5, it easily follows that

$$\mathbb{P}(\exists x \in Q(\lfloor e^{\beta(k_* -(\gamma_0/2))} \rfloor) \text{ such that } \check{\tau}_1^{\oplus}(x) > \tau_{\oplus} + T) \to 0 \tag{9.1}$$

Lemma 9.1. Let us consider $\sigma^{\eta}_{Q+x;\,t}$ with the initial condition η having no minuses. Then $\forall\, \delta > 0$

$$\mathbb{P}_{\eta}(\check{\tau}_1^{+}(x) < T \mid \mathscr{E}) \geqslant e^{-\beta(\Gamma_+ - k + \delta)} \tag{9.2}$$

Proof. Let η^+ be the ground configuration in $Q+x$ with minus boundary conditions, namely the configuration having $Q(D-1)+x$ full of pluses and $Q \setminus Q(D-1)+x$ full of zeroes.

We use the fact that the ground configuration of the largest cycle containing η and not η^+ cannot contain minuses.

On the other hand, from the analysis of the basins of attraction of the minima in Lemma 6.1, the saddle $\mathscr{S}(\underline{0}, \eta^+)$ does not contain any minus. Hence the energy barrier to overcome in order to reach η^+ is the same as in the Ising-like system $\mathscr{I}^+$ (see (2.16)), namely Γ_+.

The same reasoning can be applied to every cycle compatible with $\mathscr{E}$. In this way we estimate the largest inner resistance of the largest cycle containing $\underline{0}$ and not η^+ as $\Theta^* < \gamma_+$.

The proof is the same as the one of Lemma 7.1. ▮

Again, it is easy to show (see [OS]) that in finite volume the nucleation time and the infection time have the same asymptotic behavior:

$$\mathbb{P}(\check{\tau}_1^{+}(x) < T + \check{\tau}_1^{\oplus}(x)) \geqslant e^{-\beta(\Gamma_{\oplus} - k + \delta)} \tag{9.3}$$

The finite volume result of Lemma 9.1 is delocalized in the following Lemma, whose proof is identical to the one of Lemma 7.1.

Lemma 9.2. Let η be such that for all sites x in $\Lambda^+(e^{\beta k^+})$ the number of minuses in $Q+x$ is at most $D/3$.

Then,

$$\mathbb{P}_\eta(\min_{x \in \Lambda^+} \check{\tau}_1^+(x) < T) \to 1 \tag{9.4}$$

The following key Lemma states the irreversibility of +infection process.

Lemma 9.3. Let η be such that for all sites x in Λ^+, the number of minuses in $Q+x$ is at most $D/3$.

Let $\mathscr{E}' := \{\min_{x \in \Lambda^0}(\hat{\tau}_1^+(x)) > T\}$. Then,

$$\mathbb{P}(\mathscr{E}' \cap \mathscr{E}) \to 1 \tag{9.5}$$

Proof. The proof is the same as in Lemma 7.2.

We use the fact that even with minus boundary conditions on $Q+x$, the +infected configurations have finite probability of reaching η^+ within time 1.

In this case the energy argument of the proof of Lemma 14 in [DSch2] would work too. ∎

The following Lemmata 9.4 and 9.5 give estimates on the spreading speed of the +infection to be used in Lemma 9.6. The proofs are the same as that of Lemmata 7.3 and 7.4 once we observe that in every considered configuration there exists a downhill path substituting the minuses with zeroes and leading once more to the Ising-like case.

Lemma 9.4. Let

$$\zeta_2'(x) := \check{\tau}_1^+(x) - \min \{t : \exists y, y' \text{ with } y \neq y'; \\ \|x-y\| = \|x-y'\| = 1 \text{ and } \check{\tau}_1^+(y) \leqslant t, \check{\tau}_1^+(y') \leqslant t\} \tag{9.6}$$

be the time needed to +infect the site x, once it has two +infected neighbors.

Let

$$\mathscr{E}_2' := \{\forall x \in \Lambda^+ \zeta_2'(x) < e^{2\beta\delta}\} \tag{9.7}$$

Then, $\forall\, \delta > 0$, for large β,

$$\mathbb{P}(\mathscr{E}_2' \cap \mathscr{E}') \geqslant 1 - e^{-e^{\beta\delta}} \tag{9.8}$$

Lemma 9.5. Let us consider the sub-lattice $\mathbb{Z}'^2 := (3D\mathbb{Z})^2$ of side-length $3D$.

Let $\mathbb{A} := \{A_i\}_i$ be a partition of $\Lambda^+ \cap \mathbb{Z}'^2$ into sets A_i.

Let

$$\zeta_1' := \min_{x \in A_i} (\check{\tau}_1^+(x) - \min_{y : \|x-y\| = 1} \check{\tau}_1^+(y)) \tag{9.9}$$

$$\mathscr{E}_1'(i) := \left\{ \forall i \zeta_1'(i) < \frac{e^{\beta(\gamma_+ + 2\delta)}}{|A_i|} \right\} \tag{9.10}$$

Then, $\forall\, \delta > 0$, for large β,

$$\mathbb{P}(\mathscr{E}_1' \cap \mathscr{E}' \cap \mathscr{E}) \geqslant 1 - e^{-e^{\beta\,\delta}} \tag{9.11}$$

We can now use the shell construction of [MO] to prove Lemma 9.6 and then conclude the proof of Proposition 1 case d).

Lemma 9.6. Let $M > \frac{\gamma_+}{2}$, $L := \lfloor e^{\beta M} \rfloor$ and $k(M) := M + \frac{\gamma_+}{2}$;

Let us consider an initial configuration where for all x in Λ^+, the number of minuses in $Q + x$ is at most $D/3$ and $Q(D-1)$ is full of pluses.

Then $\forall k' > k(M)$

$$\max_{x \in Q(L)} \{\check{\tau}_1^+(x)\} < e^{\beta k'} \tag{9.12}$$

The proof is the same as the one of Lemma 7.5.

10. PROOF OF THEOREM 2)

To prove Theorem 2 we will use the following

Lemma 10.1. Let $R_{a,b}$ be a rectangle with sides not larger than $\lfloor e^{\beta a} \rfloor$ and $\lfloor e^{\beta b} \rfloor$ with $0 < b \leqslant a \leqslant k_0 - \frac{\gamma_0}{2}$ and let $\Omega_{a,b}$ be the set of all configurations having a *-cluster C of $\oplus$-infected sites touching the four sides of $R_{a,b}$. We call $I^\eta_\oplus(t)$ the set of $\oplus$-infected sites for $\sigma^{\eta}_{\Lambda^0;\,t}$.

Let

$$\tau^\eta_R := \min \{t : R_{a,b} \subset I^\eta_\oplus(t)\} \tag{10.1}$$

Then, $\forall\, \delta > 0$

$$\mathscr{E} \cap \mathscr{E}_2 \subset \{\tau^\eta_R < e^{\beta(a+\delta)}\} \tag{10.2}$$

(see (7.7) and definition before (7.4)).

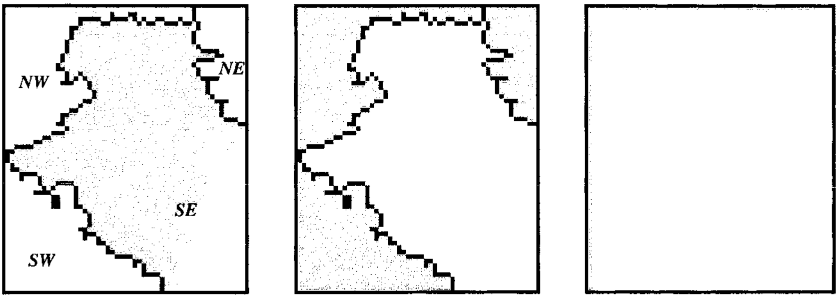


Fig. 14. The time needed to fill SE region in the left is bounded by the time needed to fill the region in the right.

Proof. Let L_N, L_E, L_S, L_W be the North, East, South and West side of the inner boundary of $R_{a,b}$, respectively and let be $P_\alpha \in \mathsf{C} \cap L_\alpha$ with $\alpha \in \{N, E, S, W\}$.

P_E and P_W are *-connected by a curve (e.g. a piece of the outer boundary of C) and so are P_N and P_S. Let ω_{EW} and ω_{NS} be such two curves. Let SE denote the region lying South of ω_{EW} and East of ω_{NS}. We define, using these two curves, SW, NW and NE in a similar way.

Let

$$\tau^{\eta}_{SE} := \inf \{t : SE \subset I^{\eta}_{\oplus}(t)\} \tag{10.3}$$

$$\Omega_{SE} := \{\rho : R_{a,b} \setminus SE \subset I^{\rho}_{\oplus}(0)\} \tag{10.4}$$

Let $\bar{\rho}$ be the configuration with $I^{\bar{\rho}}_{\oplus}(0) = L_N \cup L_W$ (see Fig. 14).

In $\mathscr{E}$ and because of the finite range of the dynamics, $\forall \rho \in \Omega_{SE}$

$$\tau^{\eta}_{SE} \leqslant \tau^{\rho}_{SE} \tag{10.5}$$

and, by using the first inequality in (2.14)

$$\tau^{\rho}_{SE} \leqslant \tau^{\bar{\rho}}_{SE} \tag{10.6}$$

Starting from $\bar{\rho}$ we fill $R_{a,b}$ "diagonally": given $i \in \mathbb{Z}$, we consider sets of the form

$$A_i := \{\underline{x} \in \mathbb{Z}^2 : x_2 = x_1 + i\} \cap R_{a,b} \tag{10.7}$$

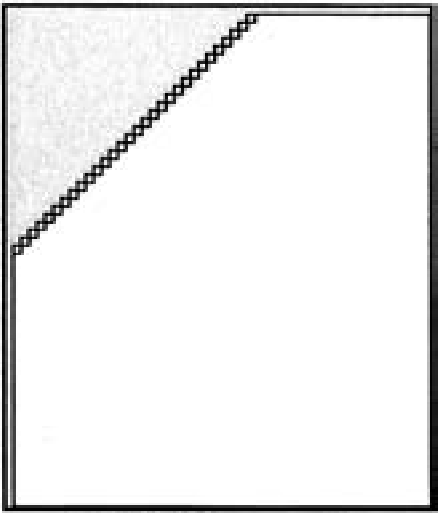


Fig. 15. Diagonal growth.

Let i' and i'' be the smallest and the largest i, respectively, such that A_i is non-empty ($i'' \leqslant i' + \lfloor e^{\beta a} \rfloor - 1$).

If A_{i-1} is entirely $\oplus$-infected, all sites of A_i have two $\oplus$-infected neighbors and can be $\oplus$-infected at rate 1. Thus, taking the sum over i of the time needed to $\oplus$-infect A_i, corresponds to $\oplus$-infect $R_{a,b}$ by $\oplus$-infecting the A_i consecutively (see Fig. 15).

Similar constructions can be given for the other parts, so that by using the definition of $\mathscr{E}_2$,

$$\tau^{\eta}_{\mathrm{R}} \leqslant 4 \sum_{i=i'}^{i''} e^{\beta \delta} \leqslant e^{\beta(a+\delta)} \quad \blacksquare \tag{10.8}$$

Proof of Theorem 2. Let $\mathscr{E}^*$ be the event $\{\min\{\bar{\tau}^{\Lambda^0}_{>D}, \tau^{\Lambda^0}_{\mathrm{LECE}}\} < T\}$ (we recall that $T = e^{\beta k}$ with $k_+ < k < k_0$). By Lemma 6.7,

$$\mathbb{P}_{-\underline{1}}(\mathscr{E}^*) \to 1 \tag{10.9}$$

On the other hand, if the process starting from $-\underline{1}$ does not contain any cluster wider than D for any time $s < T$, by using an argument like the one in Lemma 6.5 it is easy to show that $\forall s < T$, $\sigma^{\eta^0}_{\Lambda^0; s}$ contains at most one space- time cluster whose section is larger than D.

We can use the same methods as in the proof of Proposition 1 a) and bound the growth speed of the section at time s of $\mathscr{C}^{\oplus}_{0,0}(\sigma^{\eta_0^0}_{\Lambda^0; s}, s)$ as

$$v < e^{-\beta((\gamma_0/2)-\delta)} \tag{10.10}$$

where δ is arbitrarily small.

By Lemma 7.2, we get that with probability tending to one, $\oplus$-infected sites will not disinfect within T. In particular this entails that the set of $\oplus$-infected sites is connected.

By using Lemmata 6.1, 7.3 and 7.4, we can bound from below the speed of growth of this droplet as

$$v > e^{-\beta(\gamma_0+\delta)} \tag{10.11}$$

if the side-length of the droplet is smaller than $e^{\beta(\gamma_0/2)}$ or

$$v > e^{-\beta((\gamma_0/2)+\delta)} \tag{10.12}$$

if the side-length of the droplet is larger than $e^{\beta(\gamma_0/2}$.

Let us consider the process at times $t' < t'' < t'''$, defined to be $t' := T - e^{\beta(k-r)}$, $t'' = t' + c_1 e^{\beta(k-(\gamma_0/2)+\varepsilon)}$, $t''' := t'' + c_2 e^{\beta(k_+ + \varepsilon)} = t' + c_1 e^{\beta(k-(\gamma_0/2)+\varepsilon)} + c_2 e^{\beta(k_+ + \varepsilon)}$. We will show that, with large probability, $R^{\oplus}_{ext}(t')$ is $\oplus$-infected within the time t'' and is $+$infected within the time t'''. Then we will show that $t''' < T$ for suitable ε and large β.

By basic estimates on the speed of growth (see proof of Proposition 1) $\exists c_3, c_4 > 0$ such that

$$\mathbb{P}(\mathrm{diam}(R^{\oplus}_{ext}(t')) \geqslant c_3 e^{\beta(k-(\gamma_0/2))} \mid \mathscr{E} \cap \mathscr{E}_1 \cap \mathscr{E}_2) \to 1 \tag{10.13}$$

and

$$\mathbb{P}(\mathrm{diam}(R^{\oplus}_{ext}(T)) - \mathrm{diam}(R^{\oplus}_{ext}(t')) \leqslant c_4 e^{\beta(k-(\gamma_0/2)-r)} \mid \mathscr{E}^*) \to 1 \tag{10.14}$$

By using Lemmata 7.2, 7.3 and 10.1, we can show that $\forall \varepsilon > 0$, we have at time t''

$$\mathbb{P}(\forall x \in R^{\oplus}_{ext}(t'), \check{\tau}^{\oplus}_1(x) \leqslant t'') \to 1 \tag{10.15}$$

We now tile $R^{\oplus}_{ext}(t')$ with squares of side $e^{\beta(k_+ - (\gamma_+/2))}$. The tiles are less than $e^{2\beta k}$. Using the same argument as in the proof of Proposition 1 case d), we can show that $\exists c_5, c_6 > 0$ such that $\forall \varepsilon > 0$, looking at the process at time t''', we have:

$$\mathbb{P}(\exists x \in R^{\oplus}_{ext}(t') : \check{\tau}^{+}_1(x) \leqslant t''') \leqslant c_5 e^{2\beta k} e^{-c_6 e^{\beta\varepsilon}} \to 0 \tag{10.16}$$

By choosing $\varepsilon = \frac{\hat{r}-r}{2}$ such that $t''' < T$, Theorem 2 is proven. ▮

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